DIRICHLET FORMS AND FINITE ELEMENT METHODS FOR THE SABR MODEL

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Abstract. We propose a deterministic numerical method for pricing vanilla options under the SABR stochastic volatility model, based on a finite element discretization of the Kolmogorov pricing equations via non-symmetric Dirichlet forms. Our pricing method is valid both in moderate interest rate environments and in low interest rate regimes, such as the currently prevalent ones, and is applicable under mild assumptions on parameter configurations of the process. The parabolic Kolmogorov pricing equations for the SABR model are degenerate at the origin, yielding nonstandard partial differential equations, for which conventional pricing methods — designed for non-degenerate parabolic equations— potentially break down. We derive here the appropriate analytic setup to handle the degeneracy of the model at the origin. That is, we construct an evolution triple of suitably chosen Sobolev spaces with singular weights, consisting of the domain of the SABR-Dirichlet form- its dual space- and the pivotal Hilbert space. In particular, we show well-posedness of the variational formulation of the SABR-pricing equations for vanilla and barrier options on this triple. Furthermore, we present finite element discretization scheme based on a (weighted) multiresolution wavelet approximation in space and a θ-scheme in time and provide an error analysis for the discretization.

1. Introduction

The stochastic alpha beta rho (SABR) model introduced by Hagan et. al in [38, 40] is today industry standard in interest rate markets. The model with parameters $\nu > 0$, $\beta \in [0,1]$, and $\rho \in [-1,1]$, is defined by the pair of coupled stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t &= Y_t X_t^{\beta} \mathrm{d}W_t, && X_0 = x_0 > 0, \\ \mathrm{d}Y_t &= \nu Y_t \mathrm{d}Z_t, && Y_0 = y_0 > 0, \\ \mathrm{d}\langle Z, W\rangle_t &= \rho \mathrm{d}t, && 0 \le t \le T < \infty, \end{aligned} \tag{1.1}$$

where W and Z are ρ-correlated Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$. The SABR process (X, Y) takes values in the state space $D = [0,\infty) \times (0,\infty)$, and describes the dynamics of a forward rate X with stochastic volatility Y and with the initial values $x_0 > 0$ and $y_0 > 0$. The constant elasticity of variance (CEV) parameter β determines the general shape of the volatility smile and the parameter ν (often denoted by α) governs the volatility the stochastic volatility. The first pricing formula for the SABR model (the so-called Hagan formula) proposed in [38, 40], is based on an expansion of the Black-Scholes implied volatility for an asset driven by (1.1). This tractable and easy-to-implement asymptotic expansion of the implied volatility made calibration to market data easier. This, and the model's ability to capture the shape and dynamics (when the current value $X_0 = x$ of the asset changes) of the volatility smile observed in the market are virtues of the SABR model, which soon became a benchmark in interest rates derivatives markets [5, 8, 10, 61]. The Hagan expansion is only accurate when the expansion parameter is small relative to the strike, that is when time to maturity or the volatility of volatility ν is sufficiently small. For low strike options such as in low interest rate and high volatility environments, much like the ones we are facing today, this formula can yield a negative density function for the process X in (1.1), which leads to arbitrage opportunities. Therefore, as the problem of negative densities

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and arbitrage became more prevalent, it has been addressed for example in [6, 7, 10, 27, 39] by different approaches, some suggesting modifications of the SABR model or its implied volatility expansion. The attempt of suggesting suitable modifications to the original model is an intricate challenge since the Hagan expansion is deeply embedded in the market and fits market prices closely in moderate interest rate environments. Any model that deviates from its prices in those regimes may be deemed uncompetitive. This makes such pricing techniques desirable, which are applicable to the original model in all market environments. There exist several refinements to this asymptotic formula: in [58] a correction the leading order term is proposed, and [59] provides a second-order term. In in the uncorrelated case $\rho = 0$ the exact density has been derived for the absolutely continuous part of the distribution of X on $(0, \infty)$ in [6, 31, 44] and the correlated case was approximated by a mimicking model. However, it seems that these refinements have not fully resolved the arbitrage issue near the origin. Recent results [27, 36] focus on the singular part of the distribution and suggest an explanation for the irregularities appearing at interest rates near zero; and [36] provides a means to regularize Hagan's asymptotic formula at low strikes for specific parameter configurations, based on tail asymptotics derived in [24, 35].

We propose here a numerical pricing method for the (original) SABR model (1.1) with rather mild assumptions on the parameters. It is consistently applicable in all market environments and allows for the derivation of convergence rates for the numerical approximations of option prices. The most popular numerical approximation methods which were considered so far for the SABR model (or closely related models) fall into the following classes: probabilistic methods— comprising of path simulation of the process combined with suitable (quasi-) Monte Carlo approximation— were considered in [18] for the SABR model and [1, 2, 17, 20, 43] for related models. Furthermore in [18] some difficulties of Euler methods in the context of SABR are discussed. Splitting methods— where the infinitesimal generator of the process is decomposed into suitable operators for which the pricing equations can be computed more efficiently—provide a powerful tool in terms of computation efficiency for sufficiently regular processes. Such methods are considered in [12] for a model closely related to SABR (see also [11]), and in [26] for a large class of models. However, the applicability of corresponding convergence results to the SABR model itself is not fully resolved. Among fully deterministic PDE methods are most notably finite difference methods, which were considered in [5, 49] for the modification [39] of the SABR model (1.1) and finite element methods, which were described in the context of mathematical finance in [71]. In the recent textbook [41] finite element methods have been applied to a large class of financial models—including the closely related process (1.3)—and provide a robust and flexible framework to handle the stochastic finesses of these models. In spite of this, finite element approximation methods did not appear in the context of the SABR model so far in the corresponding literature. For a broad review of simulation schemes used for the SABR model (1.1) in specific parameter regimes, see also [53] and the references therein.

Standard theory provides convergence of the above methods if the considered model satisfies certain (method-specific) regularity conditions. However in the case of the SABR model, obtaining convergence rates is non-standard for these methods: The degeneracy of the SABR Kolmogorov equation at the origin violates the assumptions needed in conventional finite difference methods and—for a range of parameters—also those of ad-hoc (i.e. unweighted) finite element methods. Path simulation of the SABR process also requires nonstandard techniques due to the degeneracy of the diffusion (1.1) at zero. Nonstandard techniques often become necessary for the numerical simulation of a stochastic differential equation, when the drift and diffusion (b and σ) do not satisfy the global Lipschitz condition

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y|, \tag{1.2}$$

for $x, y \in \mathbb{R}^n$ and a constant $C > 0$ independent from x and y, cf. [64]. The degeneracy of the SABR model (1.1) at $X = 0$, originates from failure of condition (1.2) for the CEV process $\widetilde{X}$, described for parameters $\alpha > 0$, and $\beta \in [0,1]$ by the equation

$$\mathrm{d}\widetilde{X}_t = \alpha \widetilde{X}_t^{\beta} \mathrm{d}W_t \qquad \widetilde{X}_0 = \widetilde{x}_0 > 0, \quad 0 \le t \le T < \infty. \tag{1.3}$$

Although the exact distribution of the CEV process is available [51], simulation of the full SABR model based on it can in many cases become involved and expensive. In fact, exact formulas decomposing the SABR-distribution into a CEV part and a volatility part are only available in restricted parameter regimes, see [6, 30, 44] for the absolutely continuous part and [36] for the singular part of the distribution.

A simple space transformation (see (1.5) below) makes some numerical approximation results for the CIR model (the perhaps most well-understood degenerate diffusion) applicable to certain parameter regimes of the SABR process. The CIR process

$$\mathrm{d}S_t = (\delta - \gamma S_t)\,\mathrm{d}t + a\sqrt{S_t}\mathrm{d}W_t, \qquad S_0 = s_0 > 0, \quad 0 \le t \le T < \infty, \tag{1.4}$$

with $a > 0, \delta \ge 0$ indeed reduces for the parameters $\gamma = 0$, $a = 2$ to a squared Bessel process with dimension δ on the positive real line, the connection to CEV is then made via

$$\begin{aligned} \varphi : \mathbb{R}_{\ge 0} &\longrightarrow \mathbb{R}_{\ge 0} \\ s &\longmapsto \tfrac{1}{1-\delta/2}\, s^{1-\frac{\delta}{2}}, \quad \text{where} \quad \beta = \tfrac{1-\delta}{2-\delta}, \quad \text{for} \quad \delta \neq 2, \end{aligned} \tag{1.5}$$

that is, assuming absorbing boundary conditions at zero, the law of S in (1.4) (for the parameters $\gamma = 0$, $a = 2$) under the space transformation φ in (1.5) coincides with the law of $\widetilde{X}$ in (1.3). Recent results in this direction, exploring probabilistic approximation methods for diffusions where the global Lipschitz continuity (1.2) is violated, can be found for example in [17, 20] and [43], see also [3, 25] and the references therein. Establishing strong convergence rates in the case $2\delta < a^2$ where the boundary is accessible as in [43], is of particular difficulty, as this case renders coefficients of the SDE (1.4) neither globally, nor locally Lipschitz continuous on the state space. Yet, these convergence results do not cover the parameter range of the SABR model. Further approximation schemes are presented in [1, 2] which apply to CIR processes with accessible boundary and both strong and weak convergence for the approximation are studied. The weak error analysis of Talay and Tubaro [70] yields second order convergence of the schemes proposed in [1, 2], which covers the parameters $0 < \beta < \frac{1}{2}$ but the results do not directly carry over to the case $\frac{1}{2} \le \beta < 1$.

Here, we turn to a fully deterministic numerical method based on discretizations of the Kolmogorov partial differential equations, using finite elements. We derive the appropriate analytic setup to handle the degeneracy of the model at the origin. That is, we construct a suitable evolution triple of Sobolev spaces with singular weights on which well-posedness of the variational formulation of the SABR-pricing equations holds. The proposed method for space-discretization is based on the Dirichlet form corresponding to the SABR stochastic differential equation. Specifically, using the Dirichlet form we recast the Kolmogorov pricing equations in weak (variational) form and show the so-called *well posedness* of the latter. We use the weighted multiresolution (wavelet) Galerkin discretization of [13] in the state space to approximate variational solutions of the SABR-Kolmogorov pricing equations for financial contracts (vanilla and barrier options). For the time discretization of the semigroup generated by the process (1.1) we propose a θ-scheme. We derive approximation estimates tailored to our weighted setup, measuring the error between the true solution of the pricing equations and their projection to the discretization spaces. Based on these, we conclude error estimates akin to [60] for our fully discrete scheme. Under appropriate regularity assumptions on the payoff we obtain the full convergence rate for our finite element approximation. The advantage of the presented method is that it allows for a consistent pricing with very mild parameter assumptions on the SABR process and it is robustly applicable for moderate interest rate environments as well as in the current low interest rate regimes. Furthermore, the proposed discretization can be applied to compute prices of compound options or multi-period contracts without substantial modifications of the numerical methodology, cf. [64].

The article is organized as follows. Section 2 is devoted to the formulation of the SABR pricing problem in the appropriate analytic setting and an outline of the general idea of the variational analysis underlying the proposed finite element method for the SABR model. In Section 2.1 we introduce some notations and review existing results which will be used for our finite element discretization. In Section 2.2 we introduce the SABR Dirichlet form and cast the variational formulation of the SABR pricing equation in a suitable setting. We then propose a Gelfand triplet of spaces for our finite element discretization, consisting of a space $\mathcal{V}$ of admissible functions (the domain of the SABR-Dirichlet form), its dual space $\mathcal{V}^*$ and a pivotal Hilbert space $\mathcal{H}$, containing the domain of the Dirichlet form. In Section 2.3 we prove well-posedness of the SABR-pricing problem, which yields the existence of a unique weak solution to the variational formulation of the Kolmogorov partial differential equations on these spaces. In Section 3 we present the finite element discretization of the weak solution of the equation examined in the previous sections. Section 3.1 is devoted to the space discretization, which is carried out through a spline wavelet discretization of spaces $\mathcal{V}$ and $\mathcal{H}$. We review the multiresolution spline wavelet analysis of [41, 60] (in the unweighted case) to discretize the volatility dimension. The forward dimension (the CEV part) is more delicate, due to its degeneracy at zero. In this case we apply the weighted multiresolution norm equivalences, proven in [13] which are suitable to this degeneracy. We pass from the univariate case to the bivariate case by constructing tensor products of the discretized spaces in each dimension as outlined in [41]. Finally, we specify the mass- and stiffness matrices involved in the space discretization. In Section 3.2 we present the fully discrete scheme by applying a θ-scheme in the time-stepping. We follow [60], to conclude that the stability of the θ-scheme continues to hold in the present setting of weighted Sobolev spaces. In Section 4 we derive error estimates for our finite element discretization. In Section 4.1 approximation estimates of the projection to our discretization spaces are established based on multiresolution (weighted) norm equivalences. We cast our estimates under specification of different regularity assumptions on the solution of our pricing equations. We use these estimates in Section 4.2 to derive convergence rates for our finite element discretization and conclude that under some regularity assumptions on the payoff, examined in the previous section, we obtain the full convergence rate. We remark here, that the approximation- and error estimates presented in Section 4 for the SABR model readily yield the corresponding approximation- and error estimates for the CEV model as a direct corollary. The well-posedness of variational formulation of the CEV pricing equations has been studied in [41, 64], however to the best of our knowledge, a presentation of the full error analysis thereof was not available in the corresponding literature so far.

2. Preliminaries and problem formulation

Preliminaries and Notations: the process X in (1.1) is a martingale [45, Remark 2], [42, Theorem 5.1] and [52, Theorem 3.1]. For two norms on a space $\mathcal{V}$ the notation $||\cdot||_{\mathcal{V}_1} \approx ||\cdot||_{\mathcal{V}_2}$ indicates that the norms are equivalent on $\mathcal{V}$. Function spaces of bivariate functions will be denoted in italic $(\mathcal{V}, \mathcal{H}, \dots)$ and spaces of univariate functions by $(V, H, \dots)$ accordingly. For a domain $G \subset \mathbb{R}^2$ (resp. interval $I \subset \mathbb{R}$) we denote by $\mathcal{L}^1_{loc}(G)$ (resp. $L^1_{loc}(I)$) the locally integrable functions on G (resp. I), and by $\mathcal{C}^\infty_0(G)$ (resp. $C^\infty_0(I)$) the smooth functions with compact support. Derivatives with respect to time will be denoted by $\dot{u}$, $\ddot{u}$, ... accordingly to ease notation.

2.1. General analytic setup.

We first recall here the analytic setup of finite element methods. In a general parabolic setup (see for example [55, Section 2.3] or [60, Section 2]) one considers on the finite time interval $J = (0, T)$, $T > 0$ a parabolic evolution problem of the following form: find $u \in C^{1,2}(J, \mathbb{R}^2) \cap C^0(\bar{J}, \mathbb{R}^2)$ which solves

$$\begin{aligned} \dot{u}(t,z) - Au(t,z) &= g(t,z), \quad t \in J,\ z \in \mathbb{R}^2 \\ u(0,z) &= u_0(z), \quad z \in \mathbb{R}^2. \end{aligned} \tag{2.1}$$

Here, the function g is referred to as the right hand side of (2.1) or the forcing term (cf. [41, p. 27]). In a financial context, evolution equations of the type (2.1) often arise as Kolmogorov equations and the forcing term often appears as $g = -ru$, where $r \in C^0(\mathbb{R}^2; \mathbb{R}_{\geq 0})$ denotes a deterministic risk-free interest rate. In this case, the operator A is the infinitesimal generator of a suitable price process $(Z_t)_{t\in[0,T]}$ and the solution $u \in C^{1,2}(J,\mathbb{R}^2)\cap C^0(\bar{J},\mathbb{R}^2)$ of (2.1) can be represented[1] as

$$u(t,z) = \mathbb{E}^t[\mathrm{e}^{-\int_t^T r(Z_s)ds} u_0(Z_T^z)], \quad t \in J,\ z \in \mathbb{R}^2 \tag{2.2}$$

under a (possibly non-unique) martingale measure, see [41, Theorem 8.1.3] for details.

For the discretization, the domain $\mathbb{R}^2$ in (2.1) will be localized to a bounded domain $G \subset \mathbb{R}^2$ with Lipschitz boundary, cf. [60, 64]. In what follows, all localization domains G will be rectangular and denote the range of admissible values which can be taken by the price (and volatility) process, see (2.14) below. The operator A is a linear second order operator and possibly degenerate (i.e. non-elliptic) at the boundary. The domain $\mathcal{D}(A)$ of the operator A is equipped with a norm $||\cdot||_{\mathcal{V}}$ and the completion of $\mathcal{D}(A)$ under this norm will be denoted by $\mathcal{V}$. Furthermore, we denote by $\mathcal{H}$ a separable Hilbert space—the so called pivot space—containing $\mathcal{V}$, such that $\mathcal{V} \hookrightarrow \mathcal{H}$ is a dense embedding. For degenerate diffusions, the pivot space $\mathcal{H}$ could be a weighted $\mathcal{L}^2$ space over G (such as the space $\mathcal{H}$ in (2.14) below), cf. [64]. The inner product $(\cdot,\cdot)_{\mathcal{H}}$ of $\mathcal{H}$ is extended to a duality pairing $(\cdot,\cdot)_{\mathcal{V}^*\times\mathcal{V}}$ on $\mathcal{V}^*\times\mathcal{V}$, where $\mathcal{V}^*$ denotes the dual space of $\mathcal{V}$, equipped with the corresponding dual norm $||\cdot||_{\mathcal{V}^*}$. Identifying the Hilbert space $\mathcal{H}$ with its dual $\mathcal{H}^*$ we obtain the so-called Gelfand-triplet

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^*, \tag{2.3}$$

where $\hookrightarrow$ denotes a dense embedding. Applying the duality pairing $(\cdot,\cdot)_{\mathcal{V}^*\times\mathcal{V}}$ and noting that $A \in \mathcal{L}(\mathcal{V};\mathcal{V}^*)$, we can associate with the operator A in (2.1) a bilinear form $a(\cdot,\cdot): \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ by setting

$$a(u,v) := -(Au,v)_{\mathcal{V}^*\times\mathcal{V}}, \quad u,v \in \mathcal{V}. \tag{2.4}$$

Note that we do not assume the operator A in (2.1) to be self-adjoint nor the bilinearform (2.4) to be symmetric.

Definition 2.1. The form $a(\cdot,\cdot)$ is called *continuous on* $\mathcal{V}$ if there exists a $0 < C_1 < \infty$ such that

$$\forall u,v \in \mathcal{V}: \quad |a(u,v)| \le C_1 ||u||_{\mathcal{V}} ||v||_{\mathcal{V}}. \tag{2.5}$$

Definition 2.2. The form $a(\cdot,\cdot)$ is called *coercive on* $\mathcal{V}$ if there exists a constant C_2 such that $0 < C_2 \le C_1$ and

$$\forall u,v \in \mathcal{V}: \quad |a(u,u)| \ge C_2 ||u||_{\mathcal{V}}^2. \tag{2.6}$$

Remark 2.3 (Equivalent norm on $\mathcal{V}$)**.** The continuity and coercivity properties (cf. (2.5) and (2.6)) of the bilinearform $a(\cdot,\cdot)$ allow us to define for $u \in \mathcal{V}$ a so-called *energy norm* $||\cdot||_a$

$$||\cdot||_a := a(\cdot,\cdot)^{1/2} \approx ||u||_{\mathcal{V}}. \tag{2.7}$$

That is, the energy norm induced by the bilinear form is equivalent to the norm $||\cdot||_{\mathcal{V}}$ on $\mathcal{V}$.

Definition 2.4 (Gårding inequality)**.** The form (2.4) is said to *satisfy the Gårding inequality on* $\mathcal{V}$, if there exists a constant $C_3 \ge 0$ such that

$$\forall u \in \mathcal{V} \ : \ a(u,u) \ge C_2||u||_{\mathcal{V}}^2 - C_3||u||_{\mathcal{H}}^2. \tag{2.8}$$

Remark 2.5 (Reduction to Gårding inequality)**.** If the form (2.4) satisfies the (weaker) Gårding inequality (2.8), one can arrive at the coercivity property (2.6) by the substitution $v := \mathrm{e}^{-C_3 t}u$. In case (2.8) is fulfilled for the operator A, equation (2.1) implies that v satisfies (2.6), and the operator $A + C_3 I$ is coercive:

$$\dot{v}(t,z) + (A + C_3 I)v(t,z) = \mathrm{e}^{-C_3 t} g(t,z) \quad \text{in} \quad t \in J, z \in \mathbb{R}^2. \tag{2.9}$$

The time derivative of a function u in the appropriate Bochner space is understood in the weak sense: For $u \in L^2(J;\mathcal{V})$, its weak derivative in $\dot{u} \in L^2(J,\mathcal{V}^*)\cap H^1(J;\mathcal{V}^*)$ is defined by the relation

$$\int_J (\dot{u}(t),v)_{\mathcal{V}^*\times\mathcal{V}}\ \varphi(t)dt = -\int_J (u(t),v)_{\mathcal{V}^*\times\mathcal{V}}\ \dot{\varphi}(t)dt, \tag{2.10}$$

[1] Where $\tilde{u}(t,z) := u(T-t,z)$, $t \in J,\ z \in \mathbb{R}^2$ solves the Kolmogorov backward equation (2.1) to (2.2).

for every $v \in \mathcal{V}$, $\varphi \in C_0^\infty(J)$, see [41, p. 30-31] for reference and Section A.2 for more details on the above Bochner spaces. With these preparations one can formulate the variational (or weak) framework corresponding to the evolution problem (2.1). The motivation for passing to the weak formulation is that it is often not possible to find a classical solution to the original problem i.e. if there exists no function with sufficient regularity $u \in C^{1,2}(J, \mathbb{R}^2) \cap C^0(\bar{J}, \mathbb{R}^2)$ which solves (2.1). In such cases one passes to a variational reformulation of the original problem:

Definition 2.6 (Variational formulation)**.** Consider a Gelfand triplet $\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^*$ as in (2.24). Let the initial condition be $u_0 \in \mathcal{H}$ in the pivotal Hilbert space, such as the right hand side $g \in L^2(J, \mathcal{V}^*)$ of (2.1) in the Bochner space corresponding to $\mathcal{V}^*$, where $J = (0, T)$ with $T > 0$. The weak formulation of (2.1) is as follows: Find $u \in L^2(J; \mathcal{V}) \cap H^1(J; \mathcal{V}^*)$ such that $u(0) = u_0$, and for every $v \in \mathcal{V}$, $\varphi \in C_0^\infty(J)$ it holds that

$$-\int_J (u(t), v)_{\mathcal{H}} \, \dot{\varphi}(t) dt + \int_J a(u, v) \, \varphi(t) dt = \int_J (g(t), v)_{\mathcal{V}^* \times \mathcal{V}} \, \varphi(t) dt. \tag{2.11}$$

The variational formulation allows us to consider "solutions" u of (2.1) with less regularity than classical solutions of the original problem, which solve the related problem (2.11). Solutions of the variational problem (2.11) are referred to as *weak solutions* of the original problem (2.1), since whenever these are sufficiently smooth, they coincide with the solutions of the corresponding original problem (2.1). In this (possibly degenerate) parabolic setup, there holds the following general theorem.

Theorem 2.7. *Let $\mathcal{V}$ and $\mathcal{H}$ be separable Hilbert spaces with a continuous dense embedding $\mathcal{V} \hookrightarrow \mathcal{H}$. Furthermore, let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a bilinear form satisfying the inequalities* (2.5) *and* (2.8)*. Then the corresponding abstract variational parabolic problem (see Definition 2.6 above) has a unique solution in $L^2(J; \mathcal{V}) \cap H^1(J; \mathcal{V}^*)$.*

Proof. See [50, Theorem 4.1]. □

Theorem 2.8. *Assume that $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and consider a bilinearform $a(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ associated with A via* (2.4)*. If $a(\cdot, \cdot)$ satisfies the properties* (2.5) *and* (2.6) *(resp.* (2.8)*) , then*

(1) *$A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ is an isomorphism and $||A||_{\mathcal{L}(\mathcal{V}, \mathcal{V}^*)} \le C_1$ and that $||A^{-1}||_{\mathcal{L}(\mathcal{V}^*, \mathcal{V})} \le \frac{1}{C_2}$.*
(2) *$-A$ is the infinitesimal generator of a bounded analytic C^0-semigroup $(P_t)_{t \ge 0}$ in $\mathcal{V}^*$.*
(3) *For $u_0 \in \mathcal{H}$ and $g \in L^2(J; \mathcal{V}^*)$, the unique (cf. Theorem 2.7) variational solution of the pricing equation* (2.1) *(resp.* (2.9)*) can be represented as*

$$u(t) = P_t u_0 + \int_0^t P_{t-s} g(s) ds. \tag{2.12}$$

Moreover, the following a priori estimate holds

$$||u||_{L^2(J; \mathcal{V})} + ||u'||_{L^2(J; \mathcal{V}^*)} + ||u||_{C(\bar{J}; \mathcal{H})} \le C \left(||g||_{L^2(J; \mathcal{H})} + ||u_0||_{\mathcal{H}} \right). \tag{2.13}$$

Proof. See [55, Theorem 2.3.], [60, Section 2.] and also [50]. □

In our analytic setup, the spaces $\mathcal{V}$ and $\mathcal{H}$ will typically be *weighted* Sobolev spaces

$$\begin{aligned} \mathcal{H} &:= \mathcal{L}^2_\omega(G) = \{u|_G \in \mathcal{L}^1_{loc}(\mathbb{R}^2) \mid \omega u \in L^2(\mathbb{R}^2)\}, \\ \mathcal{V} &:= \mathcal{H}^1_\omega(G) = \{u|_G \in \mathcal{L}^1_{loc}(\mathbb{R}^2) \mid \omega u, \omega_1 \partial_1 u, \omega_2 \partial_2 u \in L^2(\mathbb{R}^2); u|_{\mathbb{R}^2 \setminus G} = 0\}, \end{aligned} \tag{2.14}$$

where $G \subset \mathbb{R}^2$ is a bounded domain of the form $(0, R_x) \times (-R_y, R_y)$ for some real constants $R_x, R_y > 0$, and appropriate singular weight functions[2] $\omega, \omega_1, \omega_2$ (see [60, Section 2], [55, Section 2.2] for further reference on admissibility criteria for the weight functions).

Remark 2.9. For call and put options the error made by truncating the domain to $G \subset \mathbb{R}^2$ corresponds to approximating the option prices by a knock-out barrier options, up to the first hitting time of the boundary ∂G, see [41, Section 4.6 and Chapter 6]. A probabilistic argument to estimate the localization error was suggested by Cont and Voltchkova in [19]. In the case of the SABR model, the probability that the first hitting time of R_x resp. R_y occurs before T converges

[2]See Appendix A.1 for more details on the notation and properties of the weights $\omega, \omega_1, \omega_2$.

to zero as $R_x, R_y \to \infty$. The lower boundary however cannot be truncated to any domain (ϵ, R_x) for a positive ϵ without possibly introducing a considerable localization error, since the SABR process accumulates a positive mass at zero for every $T > 0$ whenever $\beta < 1$. For details see [36], where the mass at zero is calculated for several relevant parameter configurations.

2.2. **Analytic setting for the SABR model.** In this section we establish the triplet $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ of spaces, tailored to the SABR process on which we cast the Kolmogorov pricing equations in variational form (cf. (2.11)). We then proceed to show the well-posedness of these pricing equations on this triplet, i.e. that inequalities (2.5) and (2.8) are fulfilled. We conclude the section by proving that the bilinear form resulting from the weak formulation of the pricing equations is indeed a non-symmetric Dirichlet form corresponding to the (unique) law of the SABR process. Fix a time horizon $[0,T]$ and set $\widetilde{Y} = \log(Y)$, so that the SDE (1.1) becomes

$$\begin{aligned} dX_t &= X_t^{\beta} \exp(\widetilde{Y}_t) dW_t & X_0 &= x_0 > 0, \\ d\tilde{Y}_t &= \nu dZ_t - \tfrac{\nu^2}{2} dt & \tilde{Y}_0 &= \log y_0,\ y_0 > 0 \\ d\langle W, Z\rangle_t &= \rho dt. \end{aligned} \tag{2.15}$$

The solution to (1.1) is uniquely defined if one imposes boundary conditions at $X = 0$. It is standard to impose absorbing boundary conditions to ensure martingality of the process, and indeed, for the parameters $\beta \in [0.5, 1]$ this is the only choice. For the parameters $\beta \in [0, 0.5)$ one could alternatively choose reflecting boundary conditions (as suggested in [8]) in order to accomodate to market conditions where interest rates can become negative. The value at time $t \in J = (0,T)$ of a European-type contract[3] on (2.15) with payoff u_0 is

$$u(t,z) = \mathbb{E}\left[u_0(Z_\tau^z)\right], \quad t \in J \tag{2.16}$$

where $\tau := (T-t)$ and $Z_\tau^z := (X_\tau, \widetilde{Y}_\tau)$ is the process started at $z := (x, \tilde{y}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, with $(x, \tilde{y}) = (X_t, \tilde{Y}_t)$ $\mathbb{P}$-a.s. Then for $u \in C^{1,2}(J; \mathbb{R}_{\geq 0} \times \mathbb{R}) \cap C^0(\bar{J}; \mathbb{R}_{\geq 0} \times \mathbb{R})$ the Kolmogorov pricing equation to (2.15) is

$$\begin{aligned} \dot{u}(t,z) - Au(t,z) &= 0 \quad \text{in } J \times \mathbb{R}_{\geq 0} \times \mathbb{R}, \\ u(0,z) &= u_0(z) \quad \text{in } \mathbb{R}_{\geq 0} \times \mathbb{R} \end{aligned} \tag{2.17}$$

where the infinitesimal generator A of (2.15) reads

$$Af = \frac{x^{2\beta} e^{2\tilde{y}}}{2} \partial_{xx} f + \rho \nu x^{\beta} e^{\tilde{y}} \partial_{x\tilde{y}} f + \tfrac{1}{2}\nu^2 \partial_{\tilde{y}\tilde{y}} f - \tfrac{1}{2}\nu^2 \partial_{\tilde{y}} f \qquad \text{for } f \in C_0^2(D) \subset \mathcal{D}(A). \tag{2.18}$$

From now on we drop the tilde in the logarithmic volatility for notational convenience.

Definition 2.10. For any $\mu \in [\max\{-1, -2\beta\}, 1-2\beta]$, $\beta \in [0,1]$ let $G := [0, R_x) \times (-R_y, R_y) \subset \mathbb{R}_+ \times \mathbb{R}$, $R_x, R_y > 0$ be an open subset. On G we define the weighted space[4]

$$\mathcal{H} := \mathcal{L}^2(G, x^{\mu/2}) \quad = \{u : G \to \mathbb{R} \text{ measurable } \mid ||u||_{\mathcal{L}^2(G, x^{\mu/2})} < \infty\} \tag{2.19}$$

with $||u||^2_{\mathcal{L}^2(G, x^{\mu/2})} := (u,u)_{\mathcal{H}}$ for the bilinear form

$$(u,v)_{\mathcal{H}} := \int_G u(x,y) v(x,y)\ x^{\mu} dx dy, \quad u, v \in V. \tag{2.20}$$

Remark 2.11. Note that $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ is a Hilbert space, see Appendix A.1, Lemma A.5 and A.3.

A possible choice for the weight is $\mu = -\beta$ for all $\beta \in [0,1]$. Alternatively, one can distinguish the cases $\beta \in [\frac{1}{2}, 1)$ and $\beta \in [0, \frac{1}{2}) \cup \{1\}$ and choose $\mu = -\beta$ for $\beta \in [\frac{1}{2}, 1)$, but $\mu = 0$ for $\beta \in [0, \frac{1}{2}) \cup \{1\}$. Distinguishing the above parameter regimes has the advantage that we preserve the classical setting of an unweighted $\mathcal{L}^2(G)$-space for the parameters $\beta \in [0, \frac{1}{2}) \cup \{1\}$. The latter choice furthermore highlights, that our analytic setting consistently extends the univariate CEV case to the bivariate SABR case: see [41, Section 4.5] and Remark 2.16 for the analytic setting for CEV.

[3]We assume for notational simplicity zero risk-free interest rates here.

[4]See Section (A.1) for a reminder.

Definition 2.12. Set $G := [0, R_x) \times (-R_y, R_y) \subset \mathbb{R}_+ \times \mathbb{R}$, $R_x, R_y > 0$, the domain of interest. For the first coordinate we consider $L^2([0, R_x), x^{\mu/2}) = \{u \in L^2([0, R_x)) \text{ with } ||x^{\mu/2}u||_{L^2} < \infty\}$. On $L^2([0, R_x), x^{\mu/2})$ we define the space

$$V_x := \overline{C_0^\infty([0, R_x))}^{||\cdot||_{V_x}} \text{ where } ||u||_{V_x}^2 := ||x^{\beta+\mu/2}\partial_x u)||^2_{L^2(0,R_x)} + ||x^{\mu/2}u||^2_{L^2(0,R_x)},\ u \in C_0^\infty([0, R_x)).$$

For the second coordinate we consider on $L^2(-R_y, R_y)$ the space

$$V_y := H^1(-R_y, R_y), \text{ where } ||u||^2_{V_y} := ||\partial_y u||^2_{L^2(-R_y,R_y)} + ||u||^2_{L^2(-R_y,R_y)}, \quad u \in H^1(-R_y, R_y).$$

We define for the bivariate case

$$\mathcal{V} := \left(V_x \otimes L^2(-R_y, R_y)\right) \bigcap \left(L^2([0, R_x), x^{\mu/2}) \otimes V_y\right). \tag{2.21}$$

The dual space will be denoted by $\mathcal{V}^*$ and equipped with the usual dual norm

$$||v||_{V^*} = \sup_{u \in \mathcal{V}} \frac{(v, u)_{\mathcal{V}^* \times \mathcal{V}}}{||u||_{\mathcal{V}}}, \quad v \in \mathcal{V}^*. \tag{2.22}$$

Remark 2.13. Note that the norm on the space (2.21) is by construction[5] equivalent to

$$||u||^2_{\mathcal{V}} \approx ||x^{\beta+\mu/2}\partial_x u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}u||^2_{\mathcal{L}^2(G)}, \quad \text{for} \quad u \in \mathcal{V}. \tag{2.23}$$

Lemma 2.14. *For any $\mu \in [\max\{-1, -2\beta\}, 1 - 2\beta]$ the spaces $\mathcal{H}$, $\mathcal{V}$ and $\mathcal{V}^*$ in definitions 2.10 and 2.12 form a Gelfand triplet with dense embeddings*

$$\mathcal{V} \hookrightarrow \quad \mathcal{H} \cong \mathcal{H}^* \quad \hookrightarrow \mathcal{V}^* \tag{2.24}$$

Proof. Since all weights are contained in the A_2 Muckenhoupt class[6], it holds in particular that all weights are in $L^1_{loc}(G)$, and therefore $\mathcal{C}_0^\infty(G)$ is indeed a subset of both the weighted Sobolev space[7] $W^{1,2}(G; x^{\mu/2}; x^{\mu/2+\beta}, x^{\mu/2})$ and of $\mathcal{L}^2(G, x^{\mu/2})$, and by construction $\overline{\mathcal{C}_0^\infty(G)}^{||\cdot||_{\mathcal{V}}} \subset \mathcal{V}$. Therefore, it suffices to show $\overline{\mathcal{C}_0^\infty(G)}^{||\cdot||_{\mathcal{V}}} \hookrightarrow \mathcal{H}$. For this, one can take a route via symmetric Dirichlet forms: We define a symmetric bilinear form

$$\begin{aligned} \mathcal{E}(u, v) &:= \int\int_G x^{2\beta+\mu}e^{2y}\partial_x(u)\partial_x(v)\ dx\ dy + \int\int x^\mu \partial_y(u)\partial_y(v)\ dx\ dy \\ \mathcal{D}(\mathcal{E}) &:= \mathcal{C}_0^\infty(G). \end{aligned} \tag{2.25}$$

It follows from [67, Corollary 3.5] or [14, Proposition 1] that for any open $G \subset \mathbb{R}_{\geq 0} \times \mathbb{R}$, the bilinear form (2.25) is closable on the Hilbert space $\mathcal{L}^2(G, x^{\mu/2})$. Closability yields a dense subspace of $\mathcal{L}^2(G, x^{\mu/2})$ containing $\mathcal{C}_0^\infty(G)$, on which $(\mathcal{E}(\cdot, \cdot) + || \cdot ||^2_{\mathcal{H}})^{1/2}$-Cauchy sequences converge. This norm coincides with the norm $|| \cdot ||_{\mathcal{V}}$ in (2.23) by the construction (2.25) of $\mathcal{E}$. □

We define the SABR-bilinear form by the relation (2.4). Noting that $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, and extending the inner product $(\cdot, \cdot)_{\mathcal{H}}$ of the Hilbert space to the dual pairing $(\cdot, \cdot)_{\mathcal{V}^* \times \mathcal{V}}$ we set

$$a(u, v) := -(Au, v)_{\mathcal{V}^* \times \mathcal{V}}, \qquad u, v \in \mathcal{V},$$

where the operator A acts on $\mathcal{V}$ in the weak sense. The SABR-bilinear form is therefore

$$\begin{aligned} a(u, v) =& \tfrac{1}{2} \int\int_G x^{2\beta+\mu}e^{2y}\ \partial_x u\ \partial_x v\ dxdy + \tfrac{2\beta+\mu}{2} \int\int_G x^{2\beta+\mu-1}e^{2y}\ \partial_x u\ v\ dxdy \\ &+ \rho\nu \int\int_G x^{\beta+\mu}e^y\ \partial_x u\ \partial_y v\ dxdy + \rho\nu \int\int_G x^{\beta+\mu}e^y\ \partial_x u\ v\ dxdy \\ &+ \tfrac{\nu^2}{2} \int\int_G x^\mu\ \partial_y u\ \partial_y v\ dxdy + \tfrac{\nu^2}{2} \int\int_G x^\mu\ \partial_y u\ v\ dxdy, \qquad u, v \in \mathcal{V}, \end{aligned} \tag{2.26}$$

which is obtained from (2.4)—by the divergence theorem together with $\mathcal{V} \subset \mathcal{L}^1_{loc}(G)$—when Au, $u \in \mathcal{V}$ are interpreted as weak derivatives[8].

[5] See Appendix A.3.1 for details.

[6] See Appendix A.1, in particular Lemma A.4.

[7] See Definition A.2 in the Appendix for the notation.

[8] Multiplying $-Au$ with $v \in C_0^\infty$ and integrating gives (2.4), partial integration (cf. (2.10)) then yields (2.26).

Definition 2.15 (Variational formulation of the SABR pricing equation)**.** Let $\mathcal{V}, \mathcal{V}^*$ and $\mathcal{H}$ (resp. $\mathcal{L}^2(G, x^{\mu/2})$) be as in Definitions 2.12 and 2.10, and let the bilinearform $a(\cdot,\cdot)$ on $\mathcal{V}$ be as in (2.26). Furthermore, let $u_0 \in \mathcal{H}$ (resp. $u_0 \in \mathcal{L}^2(G, x^{\mu/2})$) and consider for a $T > 0$ the finite interval $J = (0, T)$. Then the variational formulation of the SABR pricing problem reads as follows: Find $u \in L^2(J; \mathcal{V}) \cap H^1(J; \mathcal{V}^*)$, such that $u(0) = u_0$, and for every $v \in \mathcal{V}$, $\varphi \in C_0^\infty(J)$

$$-\int_J (u(t), v)_H \ \dot{\varphi}(t) dt + \int_J a(u(t), v) \ \varphi(t) dt = \int_J (g(t), v)_{\mathcal{V}^* \times \mathcal{V}} \ \varphi(t) dt. \tag{2.27}$$

Remark 2.16 (The CEV case: $\nu = 0$)**.** In [41, 64] a corresponding analytic setup is studied for the univariate case: For the CEV model the Gelfand triplet $V \subset H \cong H^* \subset V$ in [41, 64] consists of the weighed spaces

$$H := L^2((0, R), x^{\mu/2})$$

and

$$V := \overline{C_0^\infty([0, R))}^{||\cdot||_V} \quad \text{with} \quad ||u||_V^2 := ||x^{\beta+\mu/2}\partial_x u)||_{L^2(0,R)}^2 + ||x^{\mu/2}u||_{L^2(0,R)}^2$$

such as its dual V^*, where the parameter $\mu \in [\max\{-1, -2\beta\}, 1 - 2\beta]$ is chosen as in Definitions 2.10, and 2.12. Indeed, setting $\nu = 0$, the SABR process (1.1) (resp. in (2.15)) with trivial volatility process reduces to the CEV model, and the state space G reduces to $[0, R_x) \subset \mathbb{R}$. Accordongly, for $\nu = 0$ the spaces $\mathcal{H}$, $\mathcal{V}$ $\mathcal{V}^*$ in Definitions 2.10, and 2.12 coincide with the spaces V, H and V^* above. Also the SABR bilinear form (2.26) reduces to the corresponding CEV bilinear form. Hence, our analytic setup extends the univariate setup of the CEV model consistently to the bivariate SABR-case. See [64, Equation (21)],[41, Equations (4.30) and (4.33)] such as [41, page 62] for the definitions[9] of V, H, V^* and [41, Equation (4.28)] for the CEV bilinear form.

2.3. **Well-posedness of the variational pricing equations and SABR Dirichlet forms.** In this section we will show that the triplet of spaces $\mathcal{V} \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{V}^*$ (Definitions 2.10, and 2.12) is tailored to the degeneracy of the infinitesimal generator (2.18) at zero. That is, in this setting the variational formulation of the SABR pricing equation has a unique solution in $\mathcal{V}$ (this is a consequence of Theorem 2.7 and well-posedness cf. Theorem 2.17) and hence the family $P_t = E[u_0(Z_t)]$, $t \geq 0$ is a strongly continuous contraction semigroup on the Hilbert space $\mathcal{H}$, cf. Theorems 2.8 and 2.17. Furthermore, the SABR-bilinear form (2.26) is a (non-symmetric) Dirichlet form with Domain $\mathcal{V}$ on the Hilbert space $\mathcal{H}$, cf. Theorem 2.22.
Key is the well-posedness result, Theorem 2.17. This is a direct consequence of Theorem 2.7 applied to Lemmas 2.18 and 2.19 below, which establish continuity and the Gårding inequality for the SABR Dirichlet form (2.26) on this triplet.

Theorem 2.17 (Well-posedness of the SABR pricing equation)**.** *For every configuration* $(\beta, |\rho|, \nu) \in [0, 1] \times [0, 1] \times \mathbb{R}_+$ *of the SABR parameters, which satisfy the condition* $|\rho|\nu^2 < 2$ *and for any* $x_0, y_0 > 0$*, the weak formulation* (2.27) *of the pricing equation* (2.17) *corresponding to the SABR model* (2.17) *admits a unique solution* $u \in L^2(J, \mathcal{V}) \cap H^1(J, \mathcal{V}^*)$ *for any forcing term*[10] $g \in L^2(J, \mathcal{V}^*)$ *and any* u_0 *in* $\mathcal{H}$*. The unique variational solution of the pricing equation can be represented as*

$$u(t, z) = P_t u_0(z) + \int_0^t P_{t-s} g(s) ds, \qquad t \geq 0, z \in G \tag{2.28}$$

for a strongly continuous semigroup $(P_t)_{t \geq 0}$ *on* $\mathcal{H}$ *with the infinitesimal generator* A *in* (2.18)*. Furthermore, the norm* $||\cdot||_a^2 := a(\cdot,\cdot)$ *induced by the bilinearform* (2.26) *is equivalent to* $||\cdot||_a \approx ||\cdot||_{\mathcal{V}}$.

Lemma 2.18. *The bilinear form* (2.26) *is continuous: There exists a constant* $C_1 > 0$ *such that*

$$|a(u, v)| \leq C_1 \ ||u||_{\mathcal{V}} \ ||v||_{\mathcal{V}}, \quad \textit{for all} \quad u, v \in \mathcal{V}. \tag{2.29}$$

Proof. The continuity statement (2.29) is a direct consequence of the following estimates:

[9] Note that V, H, V^* are presented here in a form which is adjusted to our current notation.
[10] See equation (2.1) in Section 2.1.

(1) $\frac{1}{2}\int\int_G x^{2\beta+\mu}e^{2y}\partial_x u\partial_x v dxdy \le \frac{1}{2}\left(||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)} + ||x^{\beta+\mu/2}e^y\partial_x v||^2_{\mathcal{L}^2(G)}\right),$

(2) By the Cauchy-Schwartz inequality,

$$\begin{aligned}&\tfrac{2\beta+\mu}{2}\int\int_G x^{2\beta+\mu-1}e^{2y}\partial_x u v dxdy\\ &\le \tfrac{2\beta+\mu}{2}\left(\int\int_G x^{2\beta+\mu-2}e^{2y}v^2 dxdy\right)^{1/2}\left(\int\int_G x^{2\beta+\mu}e^{2y}(\partial_x u)^2 dxdy\right)^{1/2},\end{aligned}$$

and an upper bound for the latter expression is derived from Hardy's inequality:

$$\begin{aligned}&\le \tfrac{2\beta+\mu}{2}\tfrac{2}{|2\beta+\mu-1|}||x^{\beta+\mu/2}e^y\partial_x v||_{\mathcal{L}^2(G)}||x^{\beta+\mu/2}e^y\partial_x u||_{\mathcal{L}^2(G)}\\ &\le \tfrac{2\beta+\mu}{2}\tfrac{2}{|2\beta+\mu-1|}\left(||x^{\beta+\mu/2}e^y\partial_x v||^2_{\mathcal{L}^2(G)} + ||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)}\right)\end{aligned}$$

(3) $\rho\nu\int\int_G x^{\beta+\mu}e^y\partial_x u\partial_y u dxdy \le |\rho\nu| \left(||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)} + \frac{\nu^2}{2}||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)}\right),$

(4) $\rho\nu\int\int_G x^{\beta+\mu}e^y\partial_x u\partial_y u dxdy \le |\rho\nu|\left(||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)} + \frac{\nu^2}{2}||x^{\mu/2}u||^2_{\mathcal{L}^2(G)}\right),$

(5) $\frac{\nu^2}{2}\int\int_G x^{\mu}\partial_y u\partial_y v dxdy \le \frac{\nu^2}{2}\left(||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}\partial_y v||_{\mathcal{L}^2(G)}\right),$

(6) $\frac{\nu^2}{2}\int\int_G x^{\mu}\partial_y u v dxdy \le 0 \le \frac{\nu^2}{2}\left(||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}v||_{\mathcal{L}^2(G)}\right).$

□

Lemma 2.19. *The bilinear form (2.26) satisfies the Gårding inequality, i.e. there exist constants $C_2 > 0$ and $C_3 \ge 0$ such that*

$$a(u,u) \ge C_2\ ||u||^2_{\mathcal{V}} - C_3||u||^2_{\mathcal{H}}, \quad \textit{for all} \quad u\in\mathcal{V}. \tag{2.30}$$

Proof. The Gårding inequality (2.30) is obtained from the following estimates:

(1) $\frac{1}{2}\int\int_G x^{2\beta+\mu}e^{2y}\partial_x u\partial_x u dxdy = \frac{1}{2}||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)}$

(2) Integration by parts with $(\partial_x u)u = \frac{1}{2}\partial_x(u^2)$ yields
$\frac{2\beta+\mu}{2}\int\int_G x^{2\beta+\mu-1}e^{2y}\frac{1}{2}\partial_x(u^2)dxdy = -\frac{2\beta+\mu}{2}(2\beta+\mu-1)\int\int_G x^{2\beta+\mu-2}e^{2y}u^2 dxdy$
The last line term is non-negative if and only if $\mu\in[\max\{-1,-2\beta\},1-2\beta]$, $\beta\in[0,1]$.

(3) $\rho\nu\int\int_G x^{\beta+\mu}e^y\partial_x u\partial_y u dxdy \ge \frac{-|\rho|\nu^3}{4}\left(\frac{1}{\delta}||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)} + \delta||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)}\right)$
for a constant $\delta > 0$.

(4) $\rho\nu\int\int_G x^{\beta+\mu}e^y\partial_x u\partial_y u dxdy \ge \frac{-|\rho|\nu^3}{4}\left(\epsilon||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)} + \frac{1}{\epsilon}||x^{\mu/2}u||^2_{\mathcal{L}^2(G)}\right),$
for a constant $\epsilon > 0$.

(5) $\frac{\nu^2}{2}\int\int_G x^{\mu}\partial_y u\partial_y u dxdy = \frac{\nu^2}{2}||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)}$

(6) $\frac{\nu^2}{2}\int\int_G x^{\mu}\partial_y u v dxdy = \frac{1}{2}\nu^2(x^{\mu/2}\partial_y u, x^{\mu/2}u) = 0$ by the (Dirichlet) boundary conditions.

Hence,

$$a(u,u) \ge \left(\tfrac{1}{2} - \tfrac{|\rho|\nu^3}{4\delta} - \tfrac{|\rho|\nu^3\epsilon}{4}\right)||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2} + \left(\tfrac{\nu^2}{2} - \tfrac{|\rho|\nu^3\delta}{4}\right)||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2} - \tfrac{|\rho|\nu^3}{4\epsilon}||x^{\mu/2}u||^2_{\mathcal{L}^2},$$

which yields the inequality (2.30)

$$\begin{aligned}a(u,u) &\ge C_2\left(||x^{\beta+\mu/2}e^y\partial_x u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}\partial_y u||^2_{\mathcal{L}^2(G)} + ||x^{\mu/2}u||^2_{\mathcal{L}^2(G)}\right) - C_3||x^{\mu/2}u||^2_{\mathcal{L}^2(G)}\\ &= C_2||u||^2_{\mathcal{V}} - C_3||u||^2_{\mathcal{H}},\end{aligned}$$

with $C_2 := \min\{\frac{\nu^2}{2} - \frac{|\rho|\nu^3\delta}{4}, \frac{1}{2} - \frac{|\rho|\nu^3}{4\delta} - \frac{|\rho|\nu^3\epsilon}{4}\}$ and $C_3 := C_2 + \frac{|\rho|\nu^3}{4\epsilon}$. □

It remains to verify that the constants δ and ϵ can in fact be chosen accordingly such that $C_2 > 0$ and $C_3 > 0$ hold simultaneously.

Lemma 2.20. *For every configuration $(\beta, |\rho|, \nu) \in [0,1] \times [0,1] \times \mathbb{R}_+$ of the SABR parameters, which satisfy the condition $|\rho|\nu^2 < 2$ there exist constants $\epsilon > 0$ and $\delta > 0$ such that $C_2 > 0$.*

Remark 2.21 (Discussion of the parameter restrictions)**.** Note that the case $\rho = 0, \nu > 0$ readily yields $C_2 > 0$ and for $\nu = 0$ yields the CEV model, as discussed in Remark 2.16. Furthermore, the condition on the parameters in Lemma 2.20 is satisfied for any $|\rho| \in (0,1]$ for example if $0 < \nu < \sqrt{2}$. The latter condition on ν is fulfilled in most practical scenarios observed in the market as usual values of this parameter are well below $\sqrt{2}$: The volatility of volatility typically calibrates to values around $\nu = 0.2;\ 0.4;\ 0.6$, see for example [38, 58, 62].

Proof of Lemma 2.20. For any parameter configuration with $|\rho|\nu^2 < 2$ one can choose the constant δ in such a way that

$$\tfrac{2}{|\rho|\nu} > \delta > 0, \tag{2.31}$$

and the constants ϵ accordingly, such that

$$\tfrac{2}{\nu^3|\rho|} - \tfrac{1}{\delta} > \epsilon > 0. \tag{2.32}$$

If the inequalities (2.32) and (2.32) are satisfied then $C_2 > 0$ follows. It remains to verify that (2.32) poses no contradiction to (2.31). The bounds on δ are

$$\delta \in \left(\tfrac{|\rho|\nu^3}{2}, \tfrac{2}{|\rho|\nu}\right). \tag{2.33}$$

Indeed if $|\rho|\nu^2 < 2$, then the set in (2.33) is nonempty, and there exists an ϵ satisfying (2.32). ☐

Theorem 2.22 (A non-symmetric SABR Dirichlet form)**.** *Let the bilinearform $a(\cdot,\cdot)$ be as in* (2.26) *and its domain $\mathcal{V}$ as in* (2.21)*. Then the pair $(a(\cdot,\cdot), \mathcal{V})$ is a (non-symmetric) Dirichlet form on the Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ in* (2.19)*, for every parameter configuration $(\beta, |\rho|, \nu) \in [0,1] \times [0,1] \times \mathbb{R}_+$ with $|\rho|\nu^2 < 2$, and for any $\mu \in [\max\{-1, -2\beta\}, 1 - 2\beta]$.*

Proof. The crucial statement in the above theorem, is that the pair $(a, \mathcal{V})$ is a *coercive closed form* on the Hilbert spaces $\mathcal{H}$ and $\mathcal{L}^2(G, x^{\mu/2})$ for any $\mu \in [\max\{-1, -2\beta\}, 1 - 2\beta]$, where a denotes SABR-bilinearform (2.26) and $\mathcal{V}$ is the space (2.21). Closability of $(a, \mathcal{C}_0^\infty(G))$ in $\mathcal{H}$ and $\mathcal{L}^2(G, x^{\mu/2})$ is inherited from the closability of the auxiliary symmetric form $\mathcal{E}$ in (2.25) via the equivalence of norms $a(\cdot,\cdot) \approx ||\cdot||^2_{\mathcal{V}}$, see [66, Section 3, Proposition 3.5] . The latter equivalence is a direct consequence of the continuity property (2.29) and Gårding inequality (2.30), which were proven for the form $a(\cdot,\cdot)$ on the triplets $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ such as $\mathcal{V} \subset \mathcal{L}^2(G, x^{\mu/2}) \subset \mathcal{V}^*$ in Section 2.3. Hence, in the inequality (2.29) the norm $||\cdot||_{\mathcal{V}}$ on the right hand side can be replaced by $(a(\cdot,\cdot))^{1/2}$, yielding the strong (resp. weak) sector condition (see (B.1) and Remark B.3) for the SABR-bilinearform: There exists a $C_1 > 0$, such that for all $u, v \in \mathcal{V}$

$$|a(u,v)| \leq C_1 \ (a(u,u))^{1/2} (a(v,v))^{1/2} \quad \text{resp.} \quad |a_1(u,v)| \leq C_1 \ (a_1(u,u))^{1/2} (a_1(v,v)))^{1/2}$$

for $a_1(u,v) := a(u,v) + (u,v)_{\mathcal{H}}$, $u, v \in \mathcal{V}$. cf. [66, Equations (2.4) resp. (2.3)]. Hence, $(a(\cdot,\cdot), \mathcal{V})$ is a *coercive closed form* on $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ and on $(\mathcal{L}^2(G, x^{\mu/2}), (\cdot,\cdot)_{\mathcal{H}}))$, in the sense of [66, Definition 2.4], see Definition B.2 below. The remaining contraction properties in order for the coercive closed form $(a(\cdot,\cdot), \mathcal{V})$ to be a Dirichlet form (cf. Definition B.4) follow via Theorem B.7 from the respective contraction properties of the unique (cf. Theorem B.5) semigroups on $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ and $(\mathcal{L}^2(G, x^{\mu/2}), (\cdot,\cdot)_{\mathcal{H}}))$ associated to $(a(\cdot,\cdot), \mathcal{V})$. Alternatively, the contraction properties in Definition B.2 can be shown for the bilinear form (2.26) directly: Note that for any $u \in \mathcal{V}$ it holds that $u^+ \wedge 1 \in \mathcal{V}$ (since derivatives are taken in the weak sense) and the the contraction properties are by non-negativity of the form equivalent to

$$\begin{aligned} &a(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0 \qquad \text{if and only if} \qquad a(u^+ \wedge 1, u - u^+ \wedge 1) \geq 0, \\ &a(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0 \qquad \text{if and only if} \qquad a(u - u^+ \wedge 1, u^+ \wedge 1) \geq 0. \end{aligned}$$

Since the functions $u^+ \wedge 1$ and $u - u^+ \wedge 1$ have disjoint supports, the assertion $a(u^+ \wedge 1, u - u^+ \wedge 1) = 0$ follows directly from the construction (2.26) of the bilinearform, yielding sub-Markovianity. Hence, the form $a(u^+\wedge 1, u-u^+\wedge 1)$ is a non-symmetric Dirichlet form (as $a(u,v)\neq a(v,u)$ in general). For the sake of completeness, we included in Appendix B the properties of non-symmetric Dirichlet forms which are involved in Theorem 2.22. For a comprehensive treatment of symmetric and non-symmetric Dirichlet forms, see the monographs [15, 32] and [66] respectively. □

Remark 2.23. We remark moreover that the form (2.26) is a Dirichlet form corresponding to the unique law of the SABR process. This is a consequence of the well-posedness of the SABR martingale problem (cf. [46, page 212-214]), which follows from [46, Theorem 21.7], and [46, Lemma 21.17] by pathwise uniqueness of the solutions of (1.1) when imposing Dirichlet boundary conditions at zero.

3. Discretization

In this section we derive a suitable discretization in space and time for the variational formulation of the SABR pricing equations. We propose a multiresolution approximation inspired by the (unweighted) wavelet discretization in [60, Section 3.4]. To accommodate to the current setting, we shall rely on the weighted multiresolution analysis established in [13, Sections 5.2 and 5.3], for further reference see also [64].

3.1. **Space discretization and the semidiscrete problem.**

Given $u_0\in\mathcal{V}$ and $g\in L^2(J;\mathcal{V}^*)$, first choose an approximation $u_{(0,L)}\in\mathcal{V}_L$ of the initial data u_0, where $\mathcal{V}_L\subset\mathcal{V}$ is a finite dimensional subspace. Then the semi-discrete problem reads as follows: Find $u_L\in H^1(J;\mathcal{V}_L)$, such that

$$u_L(0)=u_{(0,L)} \tag{3.1}$$

$$(\tfrac{d}{dt}u_L,v_L)_{\mathcal{H}}+a(u_L,v_L)=(g(t),v_L)_{\mathcal{V}^*\times\mathcal{V}},\quad \forall v_L\in\mathcal{V}_L. \tag{3.2}$$

We will assume $u_{(0,L)}=P_Lu_0$, were $P_L:\mathcal{V}\to\mathcal{V}_L$, $u\longmapsto u_L$ is an appropriate projector (see equation (3.17) below). The semi-discrete problem is an initial value problem for $N=\dim\mathcal{V}_L$ ordinary differential equations

$$\mathbf{M}\tfrac{d}{dt}\underline{u}(t)+\mathbf{A}\underline{u}(t)=g(t),\quad \underline{u}(0)=u_0, \tag{3.3}$$

where $\underline{u}(t)$ denotes the coefficient vector of $u_L(t)$ for $t\geq 0$, and $\mathbf{M}$ and $\mathbf{A}$ denote the mass and stiffness matrix respectively with respect to some basis of the discretization space $\mathcal{V}_L$, which we will construct in the subsequent sections.

3.1.1. *Discretization spaces.* Recall that the bivariate pricing equation (2.17) is parabolic with degenerate elliptic operator A. To accommodate to the degeneracy of A, we introduced the weighted spaces $\mathcal{H}$ and $\mathcal{V}$ with weights which are singular at the boundary $x=0$ of the domain G (cf. Definitions 2.10 and 2.12). For the well-posedness of the variational formulation of these pricing equations we introduced weighted spaces $\mathcal{V}$ and $\mathcal{H}$ with weights, which are singular at the boundary of the domain G. To construct finite element approximation spaces for $\mathcal{V}$ and $\mathcal{H}$ we first establish univariate approximation in each dimension separately. The approximation spaces to the (weighted) univariate spaces $V_x,\ V_y$ and $H_x,\ H_y$ are then assembled to obtain the bivariate approximation spaces to $\mathcal{V}$ and $\mathcal{H}$. From now on we will restrict ourselves without loss of generality to the unit interval $I=(0,1)$ if not stated otherwise. Our multiresolution analysis on $I\subset\mathbb{R}$ consists of a nested family of spaces

$$V^0\subset V^1\subset\ldots\subset V^{N^L+1}=V_L\subset\ldots\subset L^2(I,\omega)=:H(I,\omega), \tag{3.4}$$

with $\overline{\bigcup_{l\in\mathbb{N}}V^l}=L^2(I,\omega)$, where $L^2(I,\omega)$ denotes a (possibly weighted) space of square integrable functions on I with weight function ω. The nested family (3.4) is constructed as follows. Let $\mathcal{T}_0$ be a given partition of the unit interval $[0,1]$ and $L\in\mathbb{N}$ denote the discretization level. We assume that for any $L>0$ the family $\{T_0,\ldots,T_L\}$ is such, that for each $0<l<L$ the partition T_l is obtained from T_{l-1} by bisection of each subinterval, so that T_l has $N^l=C2^l$ subintervals, where C denotes the number of intervals on level 0. Hence, on the refinement level L of the discretization we have the mesh-width $C2^{-L}$ for some $C>0$. We define V^l as the space of piecewise linear functions on the triangulation $\{\mathcal{T}_l\}$, with zero values on the boundary. For any $l \in \{0, \dots, L\}$ the dimension of the space V^l is

$$\dim V^l = C\ 2^l =: N^l, \quad 0 \leq l < L, \quad \dim V^L = C\ 2^L =: N \tag{3.5}$$

for a constant $C > 0$. Furthermore, the codimension of each level is

$$M^l := N^{l+1} - N^l, \quad 0 \leq l < L. \tag{3.6}$$

3.1.2. *Wavelets for L^2-spaces on an interval.* For the univariate approximation spaces of Sobolev spaces[11] $H_x := L^2(I, \omega_0^x)$, $H_y := L^2(I)$ such as $V_x := H^1(I, \omega_1^x)$, $V_y := H^1(I)$ on the interval I, we recapitulate basic concepts and definitions of (bi-)orthogonal, compactly supported wavelets from [13, Section 2], [64, Section 4] and [65, Section 6.2]. We consider two-parameter wavelet systems $\{\psi_{l,k}\}$, $l = 0, \dots, \infty$, $k \in M^l$ of compactly supported functions $\psi_{l,k}$. Here the first index, l, denotes the "level" of refinement resp. resolution: wavelet functions $\psi_{l,k}$ with large values of the level index are well-localized in the sense that diam supp$(\psi_{l,k}) = \mathcal{O}(2^{-l})$. The second index, $k \in M^l$, measures the localization of wavelet $\psi_{l,k}$ within the interval I at scale l and ranges in the index set M^l. In order to achieve maximal flexibility in the construction of wavelet systems (which can be used to satisfy other requirements, such as minimizing their support size or to minimize the size of constants in norm equivalences, see (3.11),(3.12) cf. [65, Section 6.2]), we propose wavelet bases, which are biorthogonal in $L^2(I)$. These consist of a primal wavelet system $\{\psi_{l,k}\}$, $l = 0, ..., \infty$, $k \in M^l$ which is a Riesz basis of $L^2(I)$ (and which enter explicitly in the space discretizations) and a corresponding dual wavelet system $\{\widetilde{\psi}_{l,k}\}$, $l = 0, ..., \infty$, $k \in M^l$, which are not used explicitly in the algorithms, see [65, Section 6.2]. The primal wavelet bases $\psi_{k,l}$ span the finite dimensional spaces

$$V^l = \operatorname{span}\{\psi_{i,j} \mid 0 \leq i \leq l, 1 \leq j \leq M^l\}, \quad l > 0, \quad \text{that is} \quad V^l = V^{l-1} \bigoplus W^l, \tag{3.7}$$

that is, $V^l = V^{l-1} \bigoplus W^l$ inductively, where $W^l = \operatorname{span}\{\psi_{l,1}, \dots, \psi_{l,M^l}\}$. Hence,

$$V_L = \bigoplus_{0 \leq l < L} W^l, \quad \text{where} \quad W^l := \operatorname{span}\{\psi_{l,1}, \dots, \psi_{l,M^l}\} \tag{3.8}$$

and dual spaces are defined analogously via the dual wavelet system

$$\widetilde{V}^L := \bigoplus_{0 \leq l < L} \widetilde{W}^l, \quad \text{where} \quad \widetilde{W}^l := \operatorname{span}\{\widetilde{\psi}_{l,1}, \dots, \widetilde{\psi}_{l,M^l}\}.$$

Furthermore, the spaces $\bigoplus_{l=0}^{\infty} W^l$ and $\bigoplus_{l=0}^{\infty} \widetilde{W}^l$ are assumed to be dense in $L^2(I)$. To construct such spaces, we consider a multiresolution basis $\{\psi_{k,l}\}_{(k,l)}$ of $L^2(I, \omega)$ with the following properties:

(1) The basis functions ψ, $\widetilde{\psi}$ are biorthogonal in $L^2(I)$, that is
$$\int_0^1 \psi_{k,l}(x) \widetilde{\psi}_{k',l'} dx = \delta_{k,l} \delta_{k',l'}.$$
(2) The wavelets $\psi_{k,l}$ and their duals $\widetilde{\psi}_{k,l}$ are local with respect to the corresponding scale and normalized, that is
$$\text{diam supp}(\psi_{k,l}) = C_{\psi_{k,l}} 2^{-l} \quad \text{such as} \quad ||\psi_{k,l}||_{L^1} = C'_{\psi} 2^{-l/2}$$
holds, where the constants C_ψ, C'_ψ may depend on the "mother" wavelet.
(3) The primal wavelets satisfy a vanishing moment condition
$$\int_0^1 \psi_{k,l}(x)\ x^\alpha dx = 0, \quad \text{for} \quad \alpha = 0, \dots, p,$$
where p denotes the polynomial order of the wavelets, see [13, Equations (2.2),(2.4)] such as [65, Equations (6.7),(6.8)]. The dual wavelets except the ones at the endpoints satisfy
$$\int_0^1 \widetilde{\psi}_{k,l}(x)\ x^\alpha dx = 0, \quad \text{for} \quad \alpha = 0, \dots, p+1,$$

[11]For the notation of the weights, recall (2.14) and Section A.1.

while at the end points the dual wavelets satisfy only

$$\int_0^1 \widetilde{\psi}_{k,l}(x)\ x^\alpha dx = 0, \quad \text{for} \quad \alpha = 1, \dots, p+1.$$

This third condition implies that the wavelets satisfy the zero Dirichlet condition namely $\psi_{k,l}(0) = \psi_{k,l}(1) = 0$, cf. [65, Equation (6.8)].

In the stock price dimension, the weighted spaces H_x and V_x such as their wavelet bases further satisfy (cf. [13, Assumption 3.1 and 3.2])

(w1) The weight function $\omega(x)$ belongs to $W^{1,\infty}((\delta,1))$ for every $\delta > 0$ and satisfies

$$C_\omega^{-1} \le \frac{\omega(x)}{x^\alpha} \le C_\omega, \qquad C_\omega^{-1} \le \frac{\omega'(x)}{x^{\alpha-1}} \le C_\omega$$

for some $\alpha \in \mathbb{R}$ and a constant $C_\omega > 0$, which only depends on the weight function.

(w2) The boundary wavelets $\psi_{k,l}(x)$ and dual wavelets $\widetilde{\psi}_{k,l}(x)$ are denoted by the index set

$$\begin{aligned} \nabla_l^L &= \{k \in \mathbb{N}_0, \gamma - 1 \le k \le 2^l - 1, 0 \in \text{supp } \psi_{k,l}\}, \\ \widetilde{\nabla}_l^L &= \{k \in \mathbb{N}_0, \gamma - 1 \le k \le 2^l - 1, 0 \in \text{supp } \widetilde{\psi}_{k,l}\}, \end{aligned}$$

and the boundary wavelets and their duals satisfy the conditions

$$\begin{aligned} &|\psi_{k,l}(x)| \le C_\psi 2^{l/2}(2^l x)^\gamma, \\ &|(\psi_{k,l})'(x)| \le C_\psi 2^{3l/2}(2^l x)^{\gamma-1}, \quad \gamma \in \mathbb{N}_0,\ k \in \nabla_l^L \\ &|\widetilde{\psi}_{k,l}(x)| \le C_\psi 2^{l/2}(2^l x)^{\tilde{\gamma}}, \\ &|(\widetilde{\psi}_{k,l})'(x)| \le C_\psi 2^{3l/2}(2^l x)^{\tilde{\gamma}-1}, \quad \tilde{\gamma} \in \mathbb{N}_0,\ k \in \widetilde{\nabla}_l^L, \end{aligned}$$

where $\gamma, \widetilde{\gamma} \in \mathbb{N}_0$ are parameters such that $\alpha + \gamma > -\frac{1}{2}$ and $-\alpha + \tilde{\gamma} > -\frac{1}{2}$ is fulfilled for the parameter α in $(w1)$. We refer to [21] for explicit constructions.

It follows that any $v \in V$ has a representation as a series and any $v_L \in V_L$ as a linear combination

$$v_L = \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_{l,j}, \quad v_L \in V_L; \qquad v = \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} v_j^l \psi_{l,j}, \quad v \in V, \tag{3.9}$$

where $v_j^l = (v, \widetilde{\psi}_{l,j})_{L^2(I)}$, cf. [13, Equation (2.9)]. Approximations of functions in V are obtained by truncating the wavelet expansion, cf. [60, Equation (3.13)] or [41, p. 161].

Definition 3.1 (Projection Operator)**.** For a subspace V of a (possibly weighted) Hilbert space $L^2(I,\omega)$ over an interval I (cf. (3.4)) the projection to the univariate finite element discretization space $P_L : V \to V_L$ is defined by truncating the wavelet expansion at refinement level L

$$P_L v := \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_{l,j}, \quad v \in V. \tag{3.10}$$

3.1.3. *Norm equivalences.* Wavelet norm equivalences are akin to the classical Parseval relation in Fourier analysis (see [41, Section 12.1.2]) allowing to express Sobolev norms in terms of sums of its Fourier coefficients. Norm equivalences are relevant for the construction of the mass- and stiffness matrices and for approximation estimates in the error analysis, cf. [13, Equations (2.5), and (2.6)]. In the unweighted univariate case there hold the standard norm[12] equivalences

$$||u||^2_{H_0^s(I)} \approx \sum_{l=0}^{\infty} \sum_{k=0}^{2^l} 2^{2ls} |u_k^l|^2. \tag{3.11}$$

For the stock price dimension V_x we need norm equivalences in weighted spaces, for which the requirements ((3.1.2) and (3.1.2), cf. [13, Section 3]) are posed on the wavelets and their duals.

[12]With a view to error analysis, it is conventional (cf. [60, Section 3.1] and [41, Section 3.6.1]) to consider functions in V, which have additional regularity: In the unweighted univariate case one considers the classical Sobolev spaces $H_0^t(I)$, $t = 0, 1, 2$, where the subscript denotes Dirichlet boundary conditions.

Theorem 3.2 (Weighted norm equivalences). *Under the assumptions of Section 3.1.2 on the wavelet basis of the discretization spaces, there holds for any $u \in L^2(I,\omega)$ the norm equivalence of its $L^2(I,\omega)$-norm and of the discrete l^2_ω-norm of its coefficients with respect to the wavelet basis.*

$$||u||^2_{L^2(I,\omega)} \approx \sum_{l=0}^{\infty} \sum_{k=0}^{M^l} \omega^2(2^{-l}k)|u^l_k|^2 \tag{3.12}$$

Proof. See [13, Theorem 3.3], and also [13, Theorem 5.1]. □

3.1.4. *Bivariate setting.* For the bivariate case we set, similarly as above, without loss of generality $G = (0,1) \times (0,1)$. The Hilbert spaces $\mathcal{H}^t(G,\omega)$, $t = 0,1,2$ can be constructed from $H^t_x(I,\omega^x_t)$ and $H^t_y(I)$ via tensor products, see appendix A.3 for explicit constructions. We define the two-dimensional discretization spaces as the tensor product of the univariate discretization spaces for the x and y coordinates

$$\mathcal{V}_L := V_{L_x} \otimes V_{L_y}. \tag{3.13}$$

Therefore, it holds similarly to (3.7) that

$$\mathcal{V}_L = \operatorname{span}\left\{\psi_{\mathbf{l},\mathbf{k}} : 0 \le l_x, l_y \le L,\ 1 \le k_x \le M^l_x, 1 \le k_y \le M^l_y\right\}, \tag{3.14}$$

where $\psi_{\mathbf{l},\mathbf{k}}(x,y) = \psi_{(l_x,l_y),(k_x,k_y)} := \psi_{l_x,k_x}(x)\psi_{l_y,k_y}(y)$, for $(x,y) \in (0,1)\times(0,1)$. Recall from (3.7) that $W^{l_x} = \operatorname{span}\{\psi_{l_x,1},\dots,\psi_{l_x,M^{l_x}}\}$ and $W^{l_y} = \operatorname{span}\{\psi_{l_y,1},\dots,\psi_{l_y,M^{l_y}}\}$ are the corresponding complement spaces. Hence it follows from (3.8) with (3.13) directly, that in the bivariate case

$$\mathcal{V}_L = \left(\bigoplus_{0\le l_x \le L} W^{l_x}\right) \otimes \left(\bigoplus_{0\le l_y \le L} W^{l_y}\right) = \bigoplus_{0\le l_x,l_y \le L} W^{l_x} \otimes W^{l_y}. \tag{3.15}$$

Every element $u \in \mathcal{L}^2(G)$ has[13] a series representation cf. [41, Equation (13.6)]

$$u = \sum_{l_x,l_y=0}^{\infty} \sum_{k_x 1}^{M^{l_x}} \sum_{k_y=1}^{M^{l_y}} u^{\mathbf{l}}_{\mathbf{k}} \psi_{\mathbf{l},\mathbf{k}}, \qquad \text{where } \mathbf{l},\mathbf{k} = (l_x,l_y),(k_x,k_y), \tag{3.16}$$

and the projection operator $P_L : \mathcal{V} \to \mathcal{V}_L$, $u \mapsto P_L u =: u_L$ is defined as in (3.10) by truncating at level L the above series representation,

$$u_L = \sum_{l_x,l_y=0}^{L} \sum_{k_x 1}^{M^{l_x}} \sum_{k_y=1}^{M^{l_y}} u^{\mathbf{l}}_{\mathbf{k}} \psi_{\mathbf{l},\mathbf{k}}, \quad u \in V, \quad \mathbf{l},\mathbf{k} = (l_x,l_y),(k_x,k_y). \tag{3.17}$$

It is immediate from the underlying tensor product structure and from the norm-equivalences (3.11) of the one-dimensional case, that in the bivariate case there holds the norm equivalence

$$||u||^2_{\mathcal{H}^s(G)} \approx \sum_{l_x,l_y=0}^{\infty} \sum_{k_x}^{2^{l_x}} \sum_{k_x}^{2^{l_x}} \left(2^{2s_x l_x} + 2^{2s_y l_y}\right) |u^{\mathbf{l}}_{\mathbf{k}}|^2, \tag{3.18}$$

for the Sobolev spaces $H^k_0(G)$, $k = 0,1,2$, where $\mathbf{l},\mathbf{k} = (l_x,l_y),(k_x,k_y)$, as in (3.9), and $\mathbf{s} = (s_x,s_y)$ cf. [41, Equation 13.8]. For notational simplicity we stated here the unweighted version of the bivariate norm equivalence. Note however, that passing from the univariate to the bivariate case is a direct consequence of the tensor product structure (see Section A.3) and is valid both in unweighted and weighted Sobolev spaces analogously.

Now we are in a position to calculate the matrices $\mathbf{S}^x, \mathbf{B}^x$, and $\mathbf{S}^x$ such as $\mathbf{S}^y, \mathbf{B}^y$, and $\mathbf{S}^y$ as building blocks of the mass- and stiffness matrices $\mathbf{M}$ and $\mathbf{A}$ appearing in the semi-discrete problem

[13]Note that $\mathcal{H} = \mathcal{L}^2(G, x^{\mu/2}) \subset \mathcal{L}^2(G)$.

(3.3) with respect to the constructed wavelet basis $\{\psi_{\mathbf{l},\mathbf{k}}\}$: for the y-coordinate, the matrices in the appropriate weighted L^2_ω-norm read

$$\begin{aligned}
\mathbf{M}^y_{\omega_y^2} &:= \left(\int_0^1 \frac{\omega_y^2(y)\psi_{l_y,k_y}(y)\psi_{l'_y,k'_y}(y)}{\omega(2^{-l_y}k_y)\omega(2^{-l'_y}k'_y)}dy\right)_{0\le l'_y,l_y\le L;\ 0\le k'_y\le 2^{l'_y},\ 0\le k_y\le 2^{l_y}}\\
\mathbf{S}^y_{\omega_y^2} &:= \left(\int_0^1 \frac{\omega_y^2(y)\psi'_{l_y,k_y}(y)\psi'_{l'_y,k'_y}(y)}{\omega(2^{-l_y}k_y)\omega(2^{-l'_y}k'_y)}dy\right)_{0\le l'_y,l_y\le L;\ 0\le k'_y\le 2^{l'_y},\ 0\le k_y\le 2^{l_y}}\\
\mathbf{B}^y_{\omega_y^2} &:= \left(\int_0^1 \frac{\omega_y^2(y)\psi'_{l_y,k_y}(y)\psi_{l'_y,k'_y}(y)}{\omega(2^{-l_y}k_y)\omega(2^{-l'_y}k'_y)}dy\right)_{0\le l'_y,l_y\le L;\ 0\le k'_y\le 2^{l'_y},\ 0\le k_y\le 2^{l_y}},
\end{aligned}\tag{3.19}$$

and for the x-coordinate, the corresponding matrices are

$$\begin{aligned}
\mathbf{M}^x_{\omega_x^2} &:= \left(\int_0^1 \frac{\omega_x^2(x)\psi_{l_x,k_x}(x)\psi_{l'_x,k'_x}(x)}{\omega(2^{-l_x}k_x)\omega(2^{-l'_x}k'_x)}dx\right)_{0\le l'_x,l_x\le L;\ 0\le k'_x\le 2^{l'_x},\ 0\le k_x\le 2^{l_x}}\\
\mathbf{S}^x_{\omega_x^2} &:= \left(\int_0^1 \frac{\omega_x^2(x)\psi'_{l_x,k_x}(x)\psi'_{l'_x,k'_x}(x)}{\omega(2^{-l_x}k_x)\omega(2^{-l'_x}k'_x)}dx\right)_{0\le l'_x,l_x\le L;\ 0\le k'_x\le 2^{l'_x},\ 0\le k_x\le 2^{l_x}}\\
\mathbf{B}^x_{\omega_x^2} &:= \left(\int_0^1 \frac{\omega_x^2(x)\psi'_{l_x,k_x}(x)\psi_{l'_x,k'_x}(x)}{\omega(2^{-l_x}k_x)\omega(2^{-l'_x}k'_x)}dx\right)_{0\le l'_x,l_x\le L;\ 0\le k'_x\le 2^{l'_x},\ 0\le k_x\le 2^{l_x}},
\end{aligned}\tag{3.20}$$

see [13, equation (3.1)]. Hence, the stiffness matrix $\mathbf{A}$ in the semi-discrete problem (3.3) is

$$\begin{aligned}
\mathbf{A} &= \left(\mathbf{A}_{(\mathbf{l}',\mathbf{k}'),(\mathbf{l},\mathbf{k})}\right)_{0\le l'_x,l_x\le L;\ 0\le k'_x\le 2^{l'_x},\ 0\le k_x\le 2^{l_x}}\\
&:= \left(a(\psi_{\mathbf{l},\mathbf{k}},\psi_{\mathbf{l}',\mathbf{k}'})\right)_{0\le l'_x,l_x\le L;\ 0\le k'_x\le 2^{l'_x},\ 0\le k_x\le 2^{l_x}},
\end{aligned}\tag{3.21}$$

where $a(\cdot,\cdot)$ denotes the SABR-bilinear form (2.26). With respect to the weighted multiresolution basis $\{\psi_{\mathbf{l},\mathbf{k}}\psi_{\mathbf{l}',\mathbf{k}'}\}$ defined in this section, the mass matrix reads

$$\mathbf{M} = \mathbf{M}^x_{x^\mu}\otimes\mathbf{M}^y_1,\tag{3.22}$$

and the stiffness matrix $\mathbf{A}$ takes the form

$$\begin{aligned}
\mathbf{A} = &\left(\mathcal{Q}_{xx}\mathbf{S}^x_{x^{2\beta+\mu}}\otimes\mathbf{M}^y_{e^{2y}} + \mathcal{Q}_{xy}\mathbf{B}^x_{x^{\beta+\mu}}\otimes\mathbf{B}^y_{e^y} + \mathcal{Q}_{yy}\mathbf{M}^x_{x^\mu}\otimes\mathbf{S}^y_1\right)\\
&+\left(c_{x_1}\mathbf{B}^x_{x^{2\beta+\mu-1}}\otimes\mathbf{M}^y_{e^{2y}} + c_{x_2}\mathbf{B}^x_{x^{\beta+\mu}}\otimes\mathbf{M}^y_{e^y} + c_y\mathbf{M}^x_{x^\mu}\otimes\mathbf{B}^y_1\right),
\end{aligned}\tag{3.23}$$

where the coefficients are $(\mathcal{Q}_{xx},\mathcal{Q}_{yy},\mathcal{Q}_{xy}) = (\frac{1}{2},\rho\nu,\frac{\nu^2}{2})$ and $(c_{x_1},c_{x_2},c_y) = (\frac{2\beta+\mu}{2},\rho\nu,\frac{\nu^2}{2})$.

3.2. **Time discretization and the fully discrete scheme.** In this section we define a θ-scheme for our time discretization to introduce the fully discrete scheme of our finite element method. Furthermore, we verify (see Proposition 3.4 below) following [60], that the stability of the θ-scheme remains valid in the setting of weighted spaces. For this, we introduce the following dual norm for the approximation spaces:

$$||f||_* := \sup_{v_L\in V_L}\frac{(f,v_L)_{\mathcal{V}^*\times\mathcal{V}}}{||v_L||_{\mathcal{V}}},\qquad f\in V_L^*,\tag{3.24}$$

furthermore, for $T<\infty$ and $M\in\mathbb{N}$ we consider the following uniform time-step and time mesh

$$k := \tfrac{T}{M},\qquad\text{and}\qquad t^m = mk,\ m=0,\dots,M.\tag{3.25}$$

The θ-scheme for the time-discretization and the fully discrete scheme are described as follows:

Definition 3.3 (θ-scheme and the fully discrete scheme)**.** Given the initial data $u_L^0 := u_{(0,L)} = P_L u^0$, for the projector in (3.17) for $m=0,\dots,M-1$ find $u_L^{m+1}\in\mathcal{V}_L$ such that for all $v_L\in\mathcal{V}_L$:

$$\tfrac{1}{k}(u_L^{m+1}-u_L^m,v_L)_{\mathcal{V}^*\times\mathcal{V}} + a(\theta u_L^{m+1}+(1-\theta)u_L^m,v_L) = (\theta g(t^{m+1})+(1-\theta)g(t^m),v_L)_{\mathcal{V}^*\times\mathcal{V}}\tag{3.26}$$

Hence, the fully discrete finite element scheme for the SABR model reads

$$(\tfrac{1}{k}\mathbf{M}+\theta\mathbf{A})\underline{u}^{m+1} = \tfrac{1}{k}\mathbf{M}\underline{u}^m - (1-\theta)\mathbf{A}\underline{u}^m + \underline{g}^{m+\theta},\quad m=0,1,\dots,M-1,\tag{3.27}$$

where $\mathbf{M}$ denotes the mass matrix (3.22), $\mathbf{A}$ the stiffness matrix (3.23), and $\underline{u}^m$ is the coefficient matrix of u_L^m with respect to the basis of $\mathcal{V}_L$.

Proposition 3.4 (Stability of the θ-scheme)**.** *For $\frac{1}{2} \leq \theta \leq 1$ let the constants C_1 and C_2 satisfy*

$$0 < C_1 < 2, \qquad C_2 \geq \frac{1}{2 - C_1}, \tag{3.28}$$

and for $0 \leq \theta < \frac{1}{2}$ denote $\lambda_A = sup_{v_L \in V_L} \frac{||v_L||^2_{\mathcal{H}}}{||v_L||^2_*}$, and the constants C_1 and C_2 be such that

$$0 < C_1 < 2 - \sigma, \qquad C_2 \geq \frac{1+(4-C_1)\sigma}{2-\sigma-C_1} \quad \text{where} \quad \sigma := k(1-2\theta)\lambda_A < 2. \tag{3.29}$$

Then the sequence $\{u_L^m\}_{m=0}^M$ of solutions of the θ-scheme 3.26 satisfy the stability estimate

$$||u_L^M||^2_{\mathcal{H}} + C_1 k \sum_{m=0}^{M-1} ||u_k^{m+\theta}||^2_{\mathcal{H}} \leq ||u_L^0||^2_{\mathcal{H}} + C_2 k \sum_{m=0}^{M-1} ||g^{m+\theta}||^2_*. \tag{3.30}$$

The proof of this proposition is analogous to the one in [60] and is delegated to the appendix.

4. Error estimates

Let $\mathcal{V}_L$, the finite dimensional approximation space of the solution space $\mathcal{V}$ be as in Section 3. Furthermore, consider $u^m(x) := u(t^m, x)$, with t^m, $m = 0, \ldots, M$ as in (3.25) and u_L^m as in the fully discrete scheme (3.26). In this section we estimate the error

$$e_L^m := u^m(x) - u_L^m(x) = (u^m - P_L u^m) + (P_L u^m - u_L^m) =: \eta^m + \xi_L^m, \tag{4.1}$$

for the the time-points t^m, $m = 0, \ldots, M$, where $P_L : \mathcal{V} \to \mathcal{V}_L$ denotes the projection 3.10 on the finite element space via truncated wavelet expansion. In Section 4.1 we derive estimates for the error η^m, $m = 0, \ldots, M$ (approximation estimates). Section 4.2 is devoted to concluding from the results of Section 4.1 corresponding error estimates for ξ_L^m , $m = 0, \ldots, M$ and the convergence of the fully discrete scheme.

4.1. Approximation estimates.

The crucial ingredient of our error analysis is the derivation of approximation estimates which measure the error $\eta^m = (u^m - P_L u^m)$ between the true solution of the pricing equation at times t^m, $m = 0, \ldots, M$ and its projection to the discretization space in a suitably chosen norm. In the unweighted case, such approximation estimates—as in (4.2) below—in the usual $H^k(I)$ (resp. $\mathcal{H}^{\mathbf{k}}(G)$) norm, $k = 0, 1, 2$ are standard, see [41, Jackson-type estimates p.163]. With view to the ensuing error analysis we derive here (see Section 4.1.2 below) analogous estimates for weighted Sobolev norms which are suitable to the Gelfand triple $\mathcal{V} \subset H \subset \mathcal{V}^*$ constructed in Section 2.3 and the corresponding discretization spaces in Section 3.1.

4.1.1. *Approximation estimates in the unweighted case.* In the unweighted univariate case it is well-known that there exists for all $u \in H^l(I)$ with $l = 0, 1, 2$ an element u_L in the corresponding discretization space V_L, such that $u_L = P_L u$ and for $k = 0, 1$ and $l = 0, 1, 2$, $l \geq k$ it holds that

$$||u - P_L u||_{H^k(I)} \leq C 2^{-(l-k)L} ||u||_{H^l(I)}, \tag{4.2}$$

where P_L denotes the projector (3.10). The existence of such an element is provided by the norm-equivalences (3.11). The same estimates hold in the two-dimensional case $G = I \times I$, see [41, Theorem 13.1.2.], in particular the approximation rate $(2^{-L})^{(l-k)}$ only depends on the discretization level 2^{-L} and is independent from dimension of the domain G. Analogously to the univariate case, for $k_x = 0, 1$ and $l_x = 0, 1, 2$, $l_x \geq k_x$ such as for $k_y = 0, 1$ and $l_y = 0, 1, 2$, $l_y \geq k_y$, it holds that

$$||u - P_L u||_{\mathcal{H}^{\mathbf{k}}(G)} \leq \begin{cases} C 2^{-(\mathbf{l}-\mathbf{k})_* L} ||u||_{\mathcal{H}^{\mathbf{l}}(G)}, & \text{if } \mathbf{k} \neq 0 \text{ or } l_x, l_y \neq 2, \\ C 2^{-(\mathbf{l}-\mathbf{k})_* L} L^{1/2} ||u||_{\mathcal{H}^{\mathbf{l}}(G)} & \text{else,} \end{cases} \tag{4.3}$$

where we denote $(\mathbf{l} - \mathbf{k})_* := \min\{l_x - k_x, l_y - k_y\}$ and where P_L is the projection operator (3.17). The estimate (4.3) is a direct consequence of (4.2) and the tensor product construction (3.13). Analogous arguments obtain as above for bivariate norm-equivalences (3.18), cf. [41, Chapter 13].

4.1.2. *Approximation estimates in the weighted case.* In the weighted setting, the order of approximation may depend on the norms of the weighted Sobolev spaces, in which we measure the error. In accord with usual conventions[14] we consider functions in $\mathcal{V}$ with additional regularity for our error analysis in the weighted setting. For this purpose we consider weighted Sobolev spaces up to second order, $\mathcal{H}^k(G,\omega)$, $k=0,1,2$, where the weight ω is yet to be chosen suitably to our setting. We aim to establish approximation estimates of the following form: for any $u \in \mathcal{H}^k(G,\omega)$, $k=0,1,2$ there exists a constant $C>0$ such that an estimate of the following type holds

$$||u-P_L u||_{\mathcal{H}^k(G,\omega)} \leq \begin{cases} C2^{-c_\omega(\mathbf{l}-\mathbf{k})_* L}||u||_{\mathcal{H}^l(G,\omega)}, & \text{for } \mathbf{k}\neq 0 \text{ or } l_x,l_y\neq 2, \\ C2^{-c_\omega(\mathbf{l}-\mathbf{k})_* L}L^{1/2}||u||_{\mathcal{H}^l(G,\omega)} & \text{else,} \end{cases} \tag{4.4}$$

for a constant $c_\omega \in \mathbb{R}_+$, which may depend on the choice of the weights in $\mathcal{H}^k(G,\omega)$, $k=0,1,2$, where $(\mathbf{l}-\mathbf{k})_* = \min\{l_x-k_x, l_y-k_y\}$ as in (4.3). In the following, we will first prove the one-dimensional version of the approximation property in weighted spaces. More specifically, we will show analogous statements to (4.2) on the weighted Sobolev spaces $H^k(I,x^{\mu/2})$, $k=0,1,2$ in the x coordinate. For this, we pass for a function $u(t,x,y)\in L^2(J,\mathcal{V}), t\in J,\ (x,y)\in G$ to $u_y(t,x)\in L^2(J,V), t\in J,\ x\in I$, for $y\in I$ (see Appendix A.3)). Note that for the y coordinate, the unweighted setting prevails and the estimate (4.2) is valid. We then proceed to the bivariate case $G=I\times I$ by constructing tensor products of univariate multiresolution finite element spaces[15]. Analogously as in the unweighted case (cf. equation (4.3)), the minimum of the obtained one-dimensional estimates then yields the estimates of (4.4), for the bivariate case $\mathcal{H}^k(G,\omega)$, $k=0,1,2$, (see Section 4.1.1 above and [41, Section 13.1]).

Definition 4.1. Consider an interval $I=(0,R)$, $R>0$ and the weighted Sobolev spaces

$$H^k_j(I,x^{\mu/2}) := \{u: I\to\mathbb{R} \text{ measurable}:\ D^a u\in L^2(I,x^{\mu/2+a\beta j}),\ a\leq k\},\quad k=0,1,2, \tag{4.5}$$

for $j=0,1$, with the norm

$$||u||^2_{H^k_j(I,x^{\mu/2})} := \sum_{a\leq k}\int_I |D^a u(x)|^2 x^{\mu+a\beta j}\,dx,\ k=0,1,2. \tag{4.6}$$

To ease notation we shall henceforth denote $H^k_{j=0}$ by H^k and $H^k_{j=1}$ by H^k_1, $k=0,1,2$.

Remark 4.2. Note that for the spaces H and V in Remark 2.16 and for the spaces H^k_j, $k=0,1,2$, $j=0,1$ in (4.5) it holds that $H=H^0((0,R),x^{\mu/2})=H^0_1((0,R),x^{\mu/2})$ and the weighted space V satisfies $V=H^1_1((0,R),x^{\mu/2})$ and $V\supset H^1((0,R),x^{\mu/2})$ such as the estimate

$$||v||^2_V \leq C_R||v||^2_{H^1((0,R),x^{\mu/2})},\quad v\in V \tag{4.7}$$

for any finite $R>0$ and a positive constant $C_R>0$.

Proposition 4.3. *The projection operator in* (3.10) *satisfies for* $k=0,1$ *and* $l=0,1,2,\ l\geq k$ *the estimate*

$$||u-P_L u||_{H^k(I,x^{\mu/2})} \leq C2^{-(l-k)L}||u||_{H^l(I,x^{\mu/2})},\qquad u\in H^2(I,x^{\mu/2}). \tag{4.8}$$

It is immediate from Remark 4.2, that the approximation estimates (4.8) readily apply to the CEV model. Furthermore, combining estimate for the x coordinate with the unweighted estimate (4.2) for the y coordinate and taking tensor products (see Appendix A.3.1 for an explicit construction of the bivariate spaces) yields that the approximation estimate (4.3) remains valid in the (weighted) bivariate case. In contrast to this, measuring the error in the norms $||u||_{H^k_{j=1}(I,x^{\mu/2})}$, $k=0,1,2$, the approximation in the x coordinate dominates the y coordinate, see Remark 4.4.

Remark 4.4. If we consider $j=1$ in Definition 4.1, we do not assume any additional integrability requirements on our solution up to first order derivatives. In this case we obtain such (weaker)

[14] Cf. [60, Section 3.1] and [41, Section 3.6.1] and see also the unweighted case above.

[15] See Appendix A.3.1 for an explicit construction

approximation estimates where the order of approximation depends on the parameter β: the projection operator $P_L : V \to V_L$ in (3.10) satisfies for $k = 0, 1$ and $l = 0, 1, 2$, $l \geq k$ the estimate

$$||u - P_L u||_{H_1^k(I,x^{\mu/2})} \leq C 2^{-(1-\beta)(l-k)L} ||u||_{H_1^l(I,x^{\mu/2})}, \qquad u \in H_1^2(I, x^{\mu/2}). \tag{4.9}$$

A proof of this remark is delegated to the Appendix C.1.

Proof of Proposition 4.3. Let $u \in \mathcal{V}$ and consider $P_L u \in \mathcal{V}_L = \mathcal{V}^L$. Then it is immediate from (3.9) and (3.10) that

$$u - P_L(u) = \sum_{l=L+1}^{\infty} \sum_{j=1}^{2^l} u_j^l \psi_{l,j}.$$

It directly follows from the norm equivalence (3.11) and the scaling of the wavelet basis-elements to unit norm in $L^2(I)$, that the derivatives satisfy

$$\begin{aligned} ||(u - P_L u)'||^2_{L^2(I,x^{\mu/2})} &= \tilde{C} \sum_{l=L+1}^{\infty} 2^{2l} \sum_{k=0}^{2^l} \omega^2(2^{-l}k)|u_k^l|^2 \\ &\geq \tilde{C} 2^{2L} \sum_{l=0}^{\infty} \sum_{k=0}^{2^l} \omega^2(2^{-l}k)|u_k^l|^2 = \tilde{C} 2^{2L} ||u - P_L u||^2_{L^2(I,x^{\mu/2})}. \end{aligned} \tag{4.10}$$

Now with $C = \frac{1}{\tilde{C}}$ and recalling that we have set $h = 2^{-2L}$, the following relation is obtained:

$$\begin{aligned} ||u - P_L u||^2_{L^2(I,x^{\mu/2})} &\leq Ch||(u - P_L u)'||^2_{L^2(I,x^{\mu/2})} \leq Ch||u'||^2_{L^2(I,x^{\mu/2})} \\ &\leq Ch \left(||u||^2_{L^2(I,x^{\mu/2})} + ||u'||^2_{L^2(I,x^{\mu/2})} \right) = Ch||u||^2_{H^1(I,x^{\mu/2})}. \end{aligned} \tag{4.11}$$

Analogously, replacing $(u - P_L u)'$ by $(u - P_L u)''$ and $(u - P_L u)$ by $(u - P_L u)'$ in equations (4.10) and (4.11), one obtains

$$||(u - P_L u)'||^2_{L^2(I,x^{\mu/2})} \leq Ch||(u - P_L u)''||^2_{L^2(I,x^{\mu/2})} \leq Ch||u''||^2_{L^2(I,x^{\mu/2})}, \tag{4.12}$$

and adding up (4.11) and (4.12) yields:

$$\begin{aligned} ||u - P_L u||^2_{H^1(I,x^{\mu/2})} &= ||u - P_L u||^2_{L^2(I,x^{\mu/2})} + ||(u - P_L u)'||^2_{L^2(I,x^{\mu/2})} \\ &\leq Ch \textstyle\sum_{k=0}^2 ||u^{(k)}||^2_{L^2(I,x^{\mu/2})} = Ch||u||^2_{H^2(I,x^{\mu/2})}. \end{aligned} \tag{4.13}$$

Finally, concatenating (4.10) and (4.11) directly yields:

$$\begin{aligned} ||u - P_L u||^2_{L^2(I,x^{\mu/2})} &\leq Ch||(u - P_L u)'||^2_{L^2(I,x^{\mu/2})} \leq C(2^{-L})^2||(u - P_L u)''||^2_{L^2(I,x^{\mu/2})} \\ &\leq C(2^{-L})^2||u''||^2_{L^2(I,x^{\mu/2})} \leq C(2^{-L})^2||u||^2_{H^1(I,x^{\mu/2})}. \end{aligned} \tag{4.14}$$

□

4.2. **Discretization error and convergence of the finite element method.** In this section we apply the estimates of Section 4.1 to derive estimates on the discretization error (cf. equation (4.1)) and to conclude the convergence of the proposed finite element approximation of the variational solution of the SABR pricing equations. For the estimates of the discretization error we follow the (unweighted) analysis of [60, Section 5]. Corresponding proofs prevail with minor modifications and are provided—accommodated to our setting and notations—in the Appendix C for easy reference.

Lemma 4.5. *For $u \in C^1(\bar{J}; \mathcal{H}^2(G, \mathbf{a}))$, the errors ξ_L^m are the solutions of the θ-scheme: Given $\xi_L^0 := P_L u^0 - u_L^0$, for $m = 0, \ldots, M-1$ find $\xi_L^{m+1} \in V_L$ such that for all $v_L \in V_L$:*

$$\tfrac{1}{k}(\xi_L^{m+1} - \xi_L^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} + a(\theta\xi_L^{m+1} + (1-\theta)\xi_L^m, v_L) =: (r^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} \tag{4.15}$$

The proof of the above Lemma relies on the observation that the errors ξ satisfy the same PDE as the solutions u, therefore the θ schemes for ξ and for u are analogous[16]. The following

[16]See Appendix C for details, where we also included for completeness a reminder of the proof of [60, Lemma 5.1]—accommodated to our notation—which directly carries over to the present situation.

corollary is a direct consequence of Lemma 4.5 and of the the stability of the θ-scheme, established in Proposition 3.4.

Corollary 4.6. *There exist constants C_1 and C_2 independent from the discretization level L and time mesh k such that the solutions of* (4.15) *satisfy the estimate*

$$||\xi_L^M||_{\mathcal{H}}^2 + C_1\ k \sum_{m=0}^{M-1} ||\xi_L^{m+\theta}||_a^2 \leq ||\xi_N^0||_{\mathcal{H}}^2 + C_2\ k \sum_{m=0}^{M-1} ||r^m||_*^2, \tag{4.16}$$

where for any $f \in V_L^*$ $||f||_* := \sup_{v_L \in \mathcal{V}_L} \frac{(f, v_L)_{\mathcal{V}\times\mathcal{V}^*}}{||v_L||_a}$, cf. (3.24) *and where r^m, $m = 0, \ldots, M-1$ denote the weak residuals defined in equation* (4.15).

The weak residual r^m, $m = 0, \ldots, M-1$ in (4.15) can be decomposed into the following parts

$$(r^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} := (r_1^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} + (r_2^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} + a(r_3^m, v_L),$$

where the components r_1^m,r_2^m and r_3^m, $m = 0, \ldots, M$ are defined as

$$\begin{aligned}
&(r_1^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} := (\tfrac{1}{k}(u^{m+1} - u^m) - \dot{u}^{m+\theta}, v_L)_{\mathcal{V}\times\mathcal{V}^*},\\
&(r_2^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} := (\tfrac{1}{k}(P_L u^{m+1} - P_L u^m) + \tfrac{1}{k}(u^{m+1} - u^m), v_L)_{\mathcal{V}\times\mathcal{V}^*},\\
&(r_3^m, v_L)_{\mathcal{V}\times\mathcal{V}^*} := a(P_L u^{m+\theta} - u^{m+\theta}, v_L).
\end{aligned} \tag{4.17}$$

Using this decomposition facilitates the following estimates for the residuals.

Lemma 4.7 (Norm estimates for the residuals)**.** *Consider the weak residuals r^m of the θ-scheme* (4.15) *for $m = 0, \ldots, M-1$. Furthermore, assume that*

$$u \in C^1(\bar{J}; \mathcal{H}_j^2(G, x^{\mu/2})) \cap C^3(J; \mathcal{H}_j^2(G, x^{\mu/2})), \qquad j = 0, 1,$$

where $\mathcal{H}_j^k(G, x^{\mu/2})$, $k = 0, 1, 2$, $j = 0, 1$ are the spaces in (A.13)*. Then there holds the estimate*

$$\begin{aligned}
||r^m||_* \leq\ C\ &\begin{cases} k^{1/2}\left(\int_{t_m}^{t_{m+1}} ||\ddot{u}||_*^2 ds\right)^{1/2}, & \theta \in [0,1] \\ k^{3/2}\left(\int_{t_m}^{t_{m+1}} ||\dddot{u}||_*^2 ds\right)^{1/2}, & \theta = \frac{1}{2} \end{cases}\\
&+2^{-L}\left(\frac{C}{k^{1/2}}\left(\int_{t_m}^{t_{m+1}} ||\dot{u}||_{\mathcal{H}_j^1(G,x^{\mu/2})}^2 ds\right)^{1/2} +\ C||u^{m+\theta}||_{\mathcal{H}_j^2(G,x^{\mu/2})}\right),
\end{aligned} \tag{4.18}$$

A proof of this Lemma is provided in the Appendix C. With these preparations, we are in a position to prove the main result of this section:

Theorem 4.8 (Convergence of the finite element approximation: SABR)**.** *Assume that*

$$u \in C^1(\bar{J}; \mathcal{H}_j^2(G, x^{\mu/2})) \cap C^3(J; \mathcal{H}_j^2(G, x^{\mu/2})), \qquad j = 0, 1,$$

where $\mathcal{H}_j^k(G, x^{\mu/2})$, $k = 0, 1, 2$, $j = 0, 1$ are the spaces in (A.13)*, and assume further that the approximation $u_{(0,L)} \in V_L$ of the initial data is quasi optimal, that is*

$$||\xi_L^0||_{\mathcal{H}}^2 = ||u_0 - u_{(0,L)}||_{\mathcal{H}}^2 \leq C2^{-2L}||u_0||_{\mathcal{H}}^2. \tag{4.19}$$

Let $u^m(z) = u(t^m, z)$, $z \in G$ for t^m, $m = 0, \ldots M$ be as in (3.25) *let u_L^m denote the solution of the fully discrete scheme* (3.26)*, and let the approximation space $\mathcal{V}_L$ be as in Section 3. Then, the following error bounds hold:*

$$\begin{aligned}
||u^M - u_L^M||_{\mathcal{H}}^2 + k\sum_{m=0}^{M-1} ||u^{m+\theta} - u_L^{m+\theta}||_a^2 \leq &C2^{-j(1-\beta)2L} \max_{0\leq t\leq T} ||u(t)||_{\mathcal{H}_j^2}^2 + 2^{-j(1-\beta)2L}\int_0^T ||\dot{u}(s)||_{\mathcal{H}_j^1}^2 ds\\
&+ C\begin{cases} k^2 \int_0^T ||\ddot{u}(s)||_*^2 ds, & 0 \leq \theta \leq 1 \\ k^4 \int_0^T ||\dddot{u}(s)||_*^2 ds, & \theta = \frac{1}{2} \end{cases}
\end{aligned}$$

for $j = 0, 1$, where $u_L^{m+\theta} = \theta u_L^{m+1} + (1-\theta)u_L^m$, and $u^{m+\theta} = \theta u^{m+1} + (1-\theta)u^m$.

Remark 4.9 (Convergence of the finite element approximation: CEV)**.** The same estimates hold for the CEV model, when we replace in Theorem 4.8 the spaces $\mathcal{H}_j^k(G, x^{\mu/2})$, $k = 0, 1, 2$, $j = 0, 1$ and the corresponding norms by $H_j^k(I, x^{\mu/2})$, $k = 0, 1, 2$, $j = 0, 1$ in (4.5) and the norms (4.6).

Proof of Theorem 4.8, and Remark 4.9. For the proofs we follow [41, Theorem 3.6.5], and [60, Theorem 5.4.] with the appropriate adjustments. For brevity we shall prove both cases $\mathcal{H}_{j=0}$ and $\mathcal{H}_{j=1}$ together, and denote the generic convergence of order by $2^{-2Lc_\omega} := 2^{-2L\ j(1-\beta)}$ as in (4.4), where $c_\omega = 1$ for $\mathcal{H}_{j=0}$ and $c_\omega = (1-\beta)$ for $\mathcal{H}_{j=1}$. By Corollary 4.6,

$$\begin{aligned}
||e_L^M||_{\mathcal{H}}^2 &= ||u^m(x) - u_L^m(x)||_{\mathcal{H}}^2 = ||\eta^m + \xi_L^m||_{\mathcal{H}}^2 \\
&\leq 2\Big(||\eta^m||_{\mathcal{H}}^2 + k\sum_{m=0}^{M-1} ||\eta^{m+\theta}||_a^2 + ||\xi_L^m||_{\mathcal{H}}^2 + k\sum_{m=0}^{M-1} ||\xi_L^{m+\theta}||_a^2\Big).
\end{aligned}$$

This yields

$$\begin{aligned}
||e_L^M||_{\mathcal{H}}^2 + C_1 k\sum_{m=0}^{M-1} ||e^{m+\theta}||_a^2 &\leq 2\Big(||\eta^M||_{\mathcal{H}}^2 + C_1 k\sum_{m=0}^{M-1} ||\eta^{m+\theta}||_a^2 + ||\xi_L^m||_{\mathcal{H}}^2 + C_1 k\sum_{m=0}^{M-1} ||\xi_L^{m+\theta}||_a^2\Big) \\
&\leq C\Big(||\eta^M||_{\mathcal{H}}^2 + k\sum_{m=0}^{M-1} ||\eta^{m+\theta}||_a^2 + ||\xi_L^0||_{\mathcal{H}}^2 + C_2\ k\sum_{m=0}^{M-1} ||r^m||_*^2\Big),
\end{aligned}$$

and by Lemma 4.7 one can further estimate the last terms to obtain

$$\begin{aligned}
&C\Big(||\eta^M||_{\mathcal{H}}^2 + k\sum_{m=0}^{M-1} ||\eta^{m+\theta}||_a^2 + ||\xi_L^0||_{\mathcal{H}}^2 + C_2\ k\sum_{m=0}^{M-1} ||r^m||_*^2\Big) \\
&\leq C\Bigg\{||\eta^M||_{\mathcal{H}}^2 + k\sum_{m=0}^{M-1} ||\eta^{m+\theta}||_a^2 + ||\xi_L^0||_{\mathcal{H}}^2 \\
&+ C_3\sum_{m=0}^{M-1} (2^{-L})^{2c_\omega}\left(\int_{t_m}^{t_{m+1}} ||\dot u||_{\mathcal{H}_j^1}^2 ds + ||u^{m+\theta}||_{\mathcal{H}_j^2}^2 + \begin{cases} k^2\int_{t_m}^{t_{m+1}} ||\ddot u||_*^2 ds, & \theta\in[0,1] \\ k^4\int_{t_m}^{t_{m+1}} ||\dddot u||_*^2 ds, & \theta = \frac{1}{2}\end{cases}\right)\Bigg\} \\
&\leq C||\xi_L^0||_{\mathcal{H}}^2 + C(2^{-L})^{2c_\omega}\int_0^T ||\dot u||_{\mathcal{V}}^2 ds + C\max_{0\leq t\leq T}(2^{-L})^{2c_\omega}||u(t)||_{\mathcal{H}_j^2}^2 + C\begin{cases} k^2\int_0^T ||\ddot u||_*^2 ds, & \theta\in[0,1] \\ k^4\int_0^T ||\dddot u||_*^2 ds, & \theta = \frac{1}{2},\end{cases}
\end{aligned}$$

where the last step follows by $||\cdot||_a \approx ||\cdot||_{\mathcal{V}} \leq C_G||\cdot||_{\mathcal{H}_j^1(G,x^{\mu/2})}$ for a $C_G > 0$ (cf. Remark A.7) and the approximation estimates (4.8) resp. (4.9). Finally, quasi optimality (4.19) of the initial data yields the statement of the Theorem. ☐

Appendix A. Reminder on properties of considered function spaces

A.1. **Weighted Sobolev spaces.** For the sake of completeness, we include a reminder on the weighted Sobolev spaces considered in this article and some of their relevant properties and refer the reader to the monographs of [47] and [16] for full details. By a weight, we shall mean a locally integrable function ω on $\mathbb{R}^2$ such that $\omega(x) > 0$ a.e. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^2$ via integration. This measure will also denoted by ω. Thus for measurable sets $E \subset R^2$, we set $\omega(E) = \int_E \omega(x)dx$.

Definition A.1 (Weighted $\mathcal{L}^2$-space)**.** Let ω be a weight and let $G \subset \mathbb{R}^2$ be open. We define $\mathcal{L}^2(G,\omega)$ as the set of measurable functions u on G such that

$$||u||_{\mathcal{L}^2(G,\omega)}^2 = \int_G |u(x)|^2\omega(x)dx < \infty \tag{A.1}$$

Definition A.2 (Weighted Sobolev space)**.** Let $k \in \mathbb{N}$ and Let $\mathbf{a} = \{\omega_{\mathbf{a}} = \omega_{\mathbf{a}}(x), x \in G, |\mathbf{a}| \leq k\}$ be a given family of weight functions on an open set $G \subset \mathbb{R}^2$. We denote by $W^k(G,\varphi)$ the set of all functions $u \in \mathcal{L}^2(G,\omega)$ for which the weak derivatives $D^{(\mathbf{a})}u$, with $|\mathbf{a}| \leq k$, belong to $\mathcal{L}^2(G,\omega_{\mathbf{a}})$. The weighted Sobolev space $W^k(G,\varphi)$ is a normed linear space if equipped with the norm

$$||u||^2_{W^k(G,\omega)} = \sum_{|\mathbf{a}|\leq k} \int_G |D^{\mathbf{a}}u(x)|^2 \omega_{\mathbf{a}}(x)dx. \tag{A.2}$$

Remark A.3. If $\omega_\omega \in \mathcal{L}^1_{loc}(G)$ then $\mathcal{C}^\infty_0(G)$ is a subset of $W^k(G,\omega)$, and we can introduce the space $W^k_0(G,\omega)$ as the closure of $\mathcal{C}^\infty_0(G)$ with respect to the norm $W^k(G,\omega)$, see also [16, 48].

The class of A^p weights was introduced by B. Muckenhoupt (cf. [57]). A weight ω is in A^p if there exists a positive constant C such that for every ball $B \subset \mathbb{R}^2$

$$\left(\frac{1}{|B|}\int_B \omega dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}dx\right)^{p-1} \leq C. \tag{A.3}$$

Lemma A.4. *For $\omega(x) := |x|$, $x \in \mathbb{R}^2$ is in A_p if and only if $-2 < \omega < 2(p-1)$.*
For $\omega(x) := \mathrm{e}(\lambda\varphi(x)) \in A_2$, with $\varphi \in W^1(G)$ and λ is sufficiently small.

Proof. See Corollary 4.4 in [68], and Corollary 2.18 in [33] □

Lemma A.5. *If $\omega \in A^p$ then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(G,\omega) \subset L^1_{loc}(G)$ for every open set $G \subset R^n$. Therefore, weak derivatives of functions in $L^p(G,\omega)$ are well-defined. Furthermore, if $\omega \in A^p$ then $\mathcal{C}^\infty(G)$ is dense in $W^{k,p}(G,\omega)$.*

Proof. See [69, Corollary 2.1.6] and [29, Theorem 1.5]. □

The weighted Sobolev space $W^{k,p}(G,\omega)$ is the set of functions $u \in \mathcal{L}^p(G,\omega)$ with weak derivatives $D^\alpha u \in \mathcal{L}^p(G,\omega)$, $|\alpha| \leq k$. The norm of u in $W^{k,p}(G,\omega)$ is

$$||u||^p_{W^k(G,\omega)} = \sum_{|\alpha|\leq k} \int_G |D^\alpha u(x)|^p \omega(x)dx.$$

A.2. **Bochner Spaces.** Let $\mathcal{H}$ denote any Hilbert space with norm $||\cdot||_{\mathcal{H}}$. For a finite $T > 0$ consider $J := (0,T)$. Then the Bochner Space $L^2(J;\mathcal{H})$ is defined as

$$L^2(J;\mathcal{H}) := \{u : \bar{J} \to \mathcal{H} \text{ measurable, } ||u||_{L^2(J;\mathcal{H})} < \infty\},$$

with norm defined by

$$||u||_{L^2(J;\mathcal{H})} := \left(\int_J ||u(t)||^2_{\mathcal{H}}dt\right)^{1/2}, \quad u \in ||u||_{L^2(J;\mathcal{H})}. \tag{A.4}$$

We call $\dot{u}$ the weak time-derivative of a function $u \in L^2(J;\mathcal{H})$, if

$$\int_J \dot{u}(t)\varphi(t)dt = -\int_J u(t)\dot{\varphi}(t))dt, \quad \forall\varphi \in C^1_0(J). \tag{A.5}$$

With the above definitions at hand, we define for $J = (0,T)$, $T > 0$, the Bochner space

$$H^1(J;\mathcal{H}) := \{u \in L^2(J;\mathcal{H}) : \dot{u} \in L^2(J;\mathcal{H})\},$$

equipped with the norm

$$||u||_{H^1(J;\mathcal{H})} := \left(\int_J ||u||^2_{\mathcal{H}} + ||\dot{u}||^2_{\mathcal{H}}dt\right)^{1/2}. \tag{A.6}$$

Furthermore, for any $n \in \mathbb{N}_0$, we define

$$C^n(J;\mathcal{H}) := \{u : \bar{J} \to \mathcal{H}, \ u \in C^n \text{ with respect to } t\} \tag{A.7}$$

See [41, Sections 2.1 and 3.1].

A.3. **Tensor Products of Hilbert Spaces.** Let $I := (0,1)$ denote the unit interval and $G := I \times I$. See in [41, Section 13.1] that the Hilbert spaces $H^k(G)$, $k = 0,1,2$ can be constructed from $H^k(I)$ via the tensor product structure:

$$\mathcal{L}^2(G) \cong \big(L^2(I) \otimes L^2(I)\big) \tag{A.8}$$

$$\mathcal{H}^1(G) \cong \big(H^1(I) \otimes L^2(I)\big) \bigcap \big(L^2(I) \otimes H^1(I)\big) \tag{A.9}$$

$$\mathcal{H}^2(G) \cong \big(H^2(I) \otimes L^2(I)\big) \bigcap \big(H^1(I) \otimes H^1(I)\big) \bigcap \big(L^2(I) \otimes H^2(I)\big) \tag{A.10}$$

Recall from [63, Chapter II. 4] that the inner products on each of the tensor-Hilbert spaces are defined as

$$\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle_{H_1 \otimes H_2} := \langle u_1, v_1 \rangle_{H_1} \langle u_2, v_2 \rangle_{H_2}, \tag{A.11}$$

for $u_1, v_1 \in H_1$ and $u_2, v_2 \in H_2$, where H_1, H_2 stand for generic Hilbert spaces (say any of the tensor products involved in (A.8),(A.9), or (A.10) above). The inner products on the intersection spaces $\mathcal{H}^k(G)$, $k = 0,1,2$ in (A.8),(A.9) and (A.10) induced by this construction are equivalent to the usual norms on these spaces. Explicitly, for any $u \equiv u_x \otimes u_y$, and $v \equiv v_x \otimes v_y \in \mathcal{H}^k(G)$ they are given by

$$\begin{aligned}
\langle u, v \rangle_{\mathcal{L}^2(G)} &= \langle u_x, v_x \rangle_{L^2(I)} \langle u_y, v_y \rangle_{L^2(I)} \\
\langle u, v \rangle_{\mathcal{H}^1(G)} &\approx \langle u_x, v_x \rangle_{H^1(I)} \langle u_y, v_y \rangle_{L^2(I)} + \langle u_x, v_x \rangle_{L^2(I)} \langle u_y, v_y \rangle_{H^1(I)} \\
\langle u, v \rangle_{\mathcal{H}^2(G)} &\approx \langle u_x, v_x \rangle_{H^2(I)} \langle u_y, v_y \rangle_{L^2(I)} + \langle u_x, v_x \rangle_{H^1(I)} \langle u_y, v_y \rangle_{H^1(I)} \\
&\quad + \langle u_x, v_x \rangle_{L^2(I)} \langle u_y, v_y \rangle_{H^2(I)}.
\end{aligned}$$

Furthermore, to justify $u \equiv u_x \otimes u_y$, and $v \equiv v_x \otimes v_y \in \mathcal{L}^2(G)$ for $u_x, v_x \in L^2(I)$ and $u_y, v_y \in L^2(I)$ we recall the following [63, Theorem II. 10. c), Chapter II. 4]:

Theorem A.6. *Let (M_1, μ_1) and (M_2, μ_2) be measure spaces so that $L^2(M_1, \mu_1)$ and $L^2(M_2, \mu_2)$ are separable, then there is a unique isomorphism such that*

$$\begin{aligned}
L^2(M_1 \times M_1, d\mu_1 \otimes d\mu_2) &\longmapsto L^2(M_1, d\mu_1; L^2(M_2, d\mu_2)) \\
f(x,y) &\longmapsto (x \mapsto f(x, \cdot)).
\end{aligned} \tag{A.12}$$

A.3.1. *Explicit construction of the bivariate spaces in Section 4.1.2.* Consider our domain of interest $G = (0, R_x) \times (-R_y, R_y)$, $R_x, R_y > 0$ and the weighted Sobolev spaces

$$\mathcal{H}^k_j(G, x^{\mu/2}) := \{u : G \to \mathbb{R} \text{ measurable} : \ \partial^{|\mathbf{a}|}_{\mathbf{a}} u \in L^2(I, x^{\mu/2 + a_x \beta j}),\ \mathbf{a} \le k\},\ k = 0,1,2, \tag{A.13}$$

for $j = 0,1$, and a multiindex $\mathbf{a}$ with $|\mathbf{a}| = a_x + a_y$, where a_x denotes the number of derivatives in direction x and a_y in direction y. The respective norms in $\mathcal{H}^k_j(G, x^{\mu/2})$ for $j = 0$ are defined by

$$\begin{aligned}
||u||^2_{\mathcal{H}^0_{j=0}(G,x^{\mu/2})} &:= ||x^{\mu/2} u||^2_{\mathcal{L}^2} \\
||u||^2_{\mathcal{H}^1_{j=0}(G,x^{\mu/2})} &:= ||x^{\mu/2} u||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_y(u)||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_x(u)||^2_{\mathcal{L}^2} \\
||u||^2_{\mathcal{H}^2_{j=0}(G,x^{\mu/2})} &:= ||x^{\mu/2} u||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_y(u)||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_x(u)||^2_{\mathcal{L}^2} \\
&\quad + ||x^{\mu/2} \partial_{yy} u||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_{xy} u||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_{yy} u||^2_{\mathcal{L}^2},
\end{aligned} \tag{A.14}$$

and for $j = 1$ by

$$\begin{aligned}
||u||^2_{\mathcal{H}^0_{j=1}(G,x^{\mu/2})} &:= ||x^{\mu/2} u||^2_{\mathcal{L}^2} \\
||u||^2_{\mathcal{H}^1_{j=1}(G,x^{\mu/2})} &:= ||x^{\mu/2} u||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_y(u)||^2_{\mathcal{L}^2} + ||x^{\beta+\mu/2} \partial_x(u)||^2_{\mathcal{L}^2} \\
||u||^2_{\mathcal{H}^2_{j=1}(G,x^{\mu/2})} &:= ||x^{\mu/2} u||^2_{\mathcal{L}^2} + ||x^{\mu/2} \partial_y(u)||^2_{\mathcal{L}^2} + ||x^{\beta+\mu/2} \partial_x(u)||^2_{\mathcal{L}^2} \\
&\quad + ||x^{\mu/2} \partial_{yy} u||^2_{\mathcal{L}^2} + ||x^{\beta+\mu/2} \partial_{xy} u||^2_{L^2} + ||x^{2\beta+\mu/2} \partial_{yy} u||^2_{L^2}.
\end{aligned} \tag{A.15}$$

Remark A.7. Note, that similarly as in Remark 4.2, the spaces $\mathcal{H}$ and $\mathcal{V}$ in Definitions 2.10 and 2.12 and the spaces $\mathcal{H}^k_j(G)$, $k = 0,1,2$, $j = 0,1$ with norms as in (A.14) and (A.15) coincide $\mathcal{H} = \mathcal{H}^0_{j=0}(G, x^{\mu/2}) = \mathcal{H}^0_{j=1}(G, x^{\mu/2})$, and the weighted space $\mathcal{V}$ satisfies $\mathcal{V} = \mathcal{H}^1_{j=1}(G, x^{\mu/2})$ and $\mathcal{V} \supset \mathcal{H}^1_{j=0}(G, x^{\mu/2})$, furthermore, on any bounded domain G there holds the estimate

$$||v||^2_{\mathcal{V}} \le C_G ||v||^2_{\mathcal{H}^1_{j=0}(G,x^{\mu/2})}, \quad v \in \mathcal{V}, \quad C_G > 0. \tag{A.16}$$

Lemma A.8. *The spaces in* (A.13)*, $k = 0, 1, 2$, $j = 0, 1$ can be constructed as tensor products of the spaces* (4.5) *and the usual (unweighted) Sobolev spaces $H^k(G)$, $k = 0, 1, 2$, $j = 0, 1$ via* (A.8) (A.9) *and* (A.10) *as follows*

$$\begin{aligned}
&\mathcal{H}^0_j(G, x^{\mu/2}) \cong \left(H^0_j(I, x^{\mu/2}) \otimes H^0(I)\right)\\
&\mathcal{H}^1_j(G, x^{\mu/2}) \cong \left(H^1_j(I, x^{\mu/2}) \otimes H^0(I)\right) \bigcap \left(H^0_j(I, x^{\mu/2}) \otimes H^1(I)\right)\\
&\mathcal{H}^2_j(G, x^{\mu/2}) \cong \left(H^2_j(I, x^{\mu/2}) \otimes H^0(I)\right) \bigcap \left(H^1_j(I, x^{\mu/2}) \otimes H^1(I)\right) \bigcap \left(H^0_j(I, x^{\mu/2}) \otimes H^2(I)\right).
\end{aligned}$$

Appendix B. Non-symmetric Dirichlet forms

We include here a condensed reminder of some of the basic concepts on non-symmetric Dirichlet forms, which are used in previous sections, in particular in Theorem 2.22. For full details see [66].

Definition B.1 (Symmetric closed form)**.** A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *symmetric closed form* on the Hilbert space $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$, if $D(\mathcal{E})$ is a dense linear subspace $\mathcal{H}$ and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a non-negative definite symmetric bilinear form, which is closed on $\mathcal{H}$. That is, $D(\mathcal{E})$ is a complete metric space with respect to the norm $\mathcal{E}_1(\cdot, \cdot)^{1/2} := (\mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_{\mathcal{H}})^{1/2}$.

Definition B.2 (Coercive closed form)**.** A pair $(\mathcal{E}, D(\mathcal{E}))$ is called a *coercive closed form* on the Hilbert space $\mathcal{H}$ if $D(\mathcal{E})$ is a dense linear subspace of $\mathcal{H}$, and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a bilinear form such that the following two conditions hold:

- Its *symmetric part* $\widetilde{\mathcal{E}}(u, v) := \frac{1}{2}\left(\mathcal{E}(u, v) + \mathcal{E}(v, u)\right)$ is a symmetric closed form on $\mathcal{H}$.
- The pair $(\mathcal{E}, D(\mathcal{E}))$ satisfies the so-called *weak sector condition*: there exists a *continuity constant* $K > 0$ such that

$$|\mathcal{E}_1(u, v)| \le K\ \mathcal{E}_1(u, u)^{1/2}\mathcal{E}_1(v, v)^{1/2} \quad \text{for all } u, v \in D(\mathcal{E}). \tag{B.1}$$

Remark B.3. Recall the continuity property 2.5 (also considered in Section 2.3)

$$|\mathcal{E}(u, v)| \le K\ \mathcal{E}(u, u)^{1/2}\mathcal{E}(v, v)^{1/2} \quad \text{for all } u, v \in D(\mathcal{E})$$

implies the weak sector condition (B.1) above.

Definition B.4 (Dirichlet form)**.** Consider a Hilbert space $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ of the form $\mathcal{H} = \mathcal{L}^2(E, m)$, where (E, m) is a measure space. A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{L}(E, m)$ is called a *(non-symmetric) Dirichlet form*, if for all $u \in D(\mathcal{E}) \subset E$, one has the *contraction properties*

$$\begin{aligned} u^+ \wedge 1 \in D(\mathcal{E}) \quad &\text{and} \quad \mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0\\ &\text{and} \quad \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \ge 0, \end{aligned} \tag{B.2}$$

where for any $u, v : E \to \mathbb{R}$, we have set

$$u \wedge v := \inf(u, v), \quad u \vee v := \sup(u, v), \quad u^+ := u \vee 0, \quad u^- := -(u \wedge 0). \tag{B.3}$$

A coercive closed form satisfying one of the two inequalities in (B.2) is called $\frac{1}{2}$- *Dirichlet form.* If $(\mathcal{E}, D(\mathcal{E}))$ is in addition symmetric, that is $\mathcal{E} = \widetilde{\mathcal{E}}$, where $\widetilde{\mathcal{E}}$ denotes the symmetric part of $\mathcal{E}$ (recall $\widetilde{\mathcal{E}}(u, v) := \frac{1}{2}\left(\mathcal{E}(u, v) + \mathcal{E}(v, u)\right)$), then $(\mathcal{E}, D(\mathcal{E}))$ is called a *symmetric* Dirichlet form.
In the latter case, the contraction property in condition (B.2) reduces to

$$\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \le \mathcal{E}(u, u). \tag{B.4}$$

See [66, Section 4, Def. 4.5]

Theorem B.5. *Let $(\mathcal{E}, D(\mathcal{E}))$ be coercive closed form on a Hilbert space $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ with continuity constant $K > 0$. Define the domain*

$$D(A) := \{u \in D(\mathcal{E}) \mid v \mapsto \mathcal{E}(u, v) \textit{ is continuous w.r.t. } (\cdot, \cdot)^{1/2}_{\mathcal{H}} \textit{ on } D(\mathcal{E})\}. \tag{B.5}$$

For any $u \in D(A)$, let Au denote the unique element in $\mathcal{H}$ such that

$$(-Au, v) = \mathcal{E}(u, v) \textit{ for all } v \in D(\mathcal{E}). \tag{B.6}$$

Then A is the generator of the unique strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on $\mathcal{H}$ which satisfies

$$\begin{aligned} G_\alpha(\mathcal{H}), \subset D(\mathcal{E}) \quad &\textit{and} \quad \mathcal{E}(G_\alpha f, u) + \alpha(G_\alpha f, u)_\mathcal{H} = (f, u)_\mathcal{H} \\ \textit{for all} \quad &f \in \mathcal{H}, \quad u \in D(\mathcal{E}), \quad \alpha > 0. \end{aligned} \tag{B.7}$$

Furthermore, since $(\mathcal{E}, D(\mathcal{E}))$ is a coercive closed form on $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$, there exists a further unique strongly continuous contraction resolvent $(\hat{G}_\alpha)_{\alpha>0}$ on $\mathcal{H}$, which satisfies

$$\begin{aligned} \hat{G}_\alpha(\mathcal{H}) \subset D(\mathcal{E}) \quad &\textit{and} \quad (f, u)_\mathcal{H} = \mathcal{E}(u, \hat{G}_\alpha f) + \alpha(u, \hat{G}_\alpha f)_\mathcal{H} \\ \textit{for all} \quad &f \in \mathcal{H}, \quad u \in D(\mathcal{E}), \quad \alpha > 0. \end{aligned} \tag{B.8}$$

In particular, $\hat{G}_\alpha$ is the adjoint of G_α for all $\alpha > 0$. That is

$$(G_\alpha f, g)_\mathcal{H} = (f, \hat{G}_\alpha g)_\mathcal{H} \quad \textit{for all} \quad f, g \in \mathcal{H}, \tag{B.9}$$

and similarly, for the (unique) strongly continuous contraction semigroups $(P_t)_{t\geq0}, (\hat{P}_t)_{t\geq0}$ corresponding to $(G_\alpha)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ respectively it holds that

$$(P_t f, g)_\mathcal{H} = (f, \hat{P}_t g)_\mathcal{H} \quad \textit{for all} \quad f, g \in \mathcal{H}, \ t \geq 0. \tag{B.10}$$

Proof. See: [66, Theorem 2.8, Corollary 2.10 and Proposition 2.16]. □

Definition B.6 (Contraction Properties)**.** Let $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$ be Hilbert space where $\mathcal{H} = \mathcal{L}^2(E, m)$, and where (E, m) is a measure space. For any $f, g \in \mathcal{L}^2(E, m)$ we write $f \leq g$ or $f < g$ for any m-classes f, g of functions on E, if the respective inequality holds m-a.e. for corresponding representatives.

- (i) Let G be a bounded linear operator on $\mathcal{L}^2(E, m)$ with domain $D(G) = \mathcal{L}^2(E, m)$. Then G is called *sub-Markovian*, if for all $f \in \mathcal{L}^2(E, m)$ the condition $0 \leq f \leq 1$ implies $0 \leq Gf \leq 1$.
- (ii) A strongly continuous contraction semigroup $(P_t)_{t\geq0}$ resp. resolvent $(G_\alpha)_{\alpha>0}$ is called *sub-Markovian* if all P_t, $t \geq 0$ resp. αG_α, $\alpha > 0$ are sub-Markovian.
- (iii) A closed, densely defined operator A on $(\mathcal{L}^2(E, m), (\cdot,\cdot)_\mathcal{H})$ is called *Dirichlet operator* if $(Au, (u-1)^+)_\mathcal{H} \leq 0$ for all $u \in D(A) \subset E$.

See [66, Section 4, Def. 4.1].

Theorem B.7. *Consider a Hilbert space $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$ of the form $\mathcal{H} = \mathcal{L}^2(E, m)$, where (E, m) is a measure space. Let $(G_\alpha)_{\alpha>0}$ be a strongly continuous contraction resolvent on $(\mathcal{L}^2(E, m), (\cdot,\cdot)_\mathcal{H})$ with corresponding generator A and semigroup $(P_t)_{t\geq0}$. Furthermore, let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive closed form on $\mathcal{L}^2(E, m)$ with continuity constant $K > 0$ and corresponding resolvent $(G_\alpha)_{\alpha>0}$. Then the following are equivalent:*

- (i) *$(P_t)_{t\geq0}$ is sub-Markovian.*
- (ii) *A is a Dirichlet operator.*
- (iii) *$(G_\alpha)_{\alpha>0}$ is sub-Markovian.*
- (iv) *for all $u \in D(\mathcal{E})$, $u^+ \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$, that is $(\mathcal{E}, D(\mathcal{E}))$ is a $\frac{1}{2}$-Dirichlet form.*

If in the above statements the operators $(G_\alpha)_{\alpha>0}$ (resp. $(P_t)_{t\geq0}$ and A) are replaced by their adjoints $(\hat{G}_\alpha)_{\alpha>0}$ (resp. $(\hat{P}_t)_{t\geq0}$ and $\hat{A}$), then the analogous equivalences hold, where in (iv) the entries of $\mathcal{E}$ are interchanged. Hence, if (iii) (resp. (ii) or (i)) holds both for $(G_\alpha)_{\alpha>0}$ (resp. A or $(P_t)_{t\geq0}$) and its adjoint $(\hat{G}_\alpha)_{\alpha>0}$ (resp. $\hat{A}$ or $(\hat{P}_t)_{t\geq0}$), then the coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ is a (non-symmetric) Dirichlet form.

Proof. See [66, Section 4, Prop. 4.3 and Theorem 4.4]. □

Appendix C. Further Proofs

Proof of Proposition 3.4. We recall and slightly modify the proof given in [60, Proposition 4.1] to accommodate to the situation at hand. Set

$$X^m := ||u_L^m||^2_{\mathcal{H}} - ||u_L^{m+1}||^2_{\mathcal{H}} + C_2 k||g^{m+\theta}||^2_* - C_1 k||u_L^{m+\theta}||^2_a. \tag{C.1}$$

If suffices to show, $X^m \geq 0$ for $m = 0, \ldots, M-1$, then $\sum_{m=0}^{M-1} X^m \geq 0$ readily yields (3.30). In the remainder of the proof we use the following shorthand notation:

$$\begin{aligned} \vartheta &:= u_h^{m+1} - u_h^m, \\ u_L^{m+\theta} &:= \theta u_L^{m+1} + (1-\theta)u_L^m = \tfrac{1}{2}(u_h^{m+1} - u_h^m) + (\theta - \tfrac{1}{2})\vartheta, \\ g^{m+\theta} &:= \theta g(t^{m+1}) + (1-\theta)g(t^m). \end{aligned} \tag{C.2}$$

It holds that

$$||u_L^{m+1}||^2_{\mathcal{H}} - ||u_L^m||^2_{\mathcal{H}} = (u_L^{m+1} - u_L^m, u_L^{m+1} + u_L^m)_{\mathcal{H}} = 2(\vartheta, u_L^{m+\theta})_{\mathcal{H}} - (2\theta - 1)(\vartheta, \vartheta)_{\mathcal{H}} \tag{C.3}$$

On the other hand, by the definition of the θ scheme (3.26)

$$\begin{aligned} 2(\vartheta, u_L^{m+\theta}) &= 2k\left(-||u_L^{m+\theta}||^2_a + (g^{m+\theta}, u_L^{m+\theta})\right) \\ &\leq 2k\left(-||u_L^{m+\theta}||^2_a + ||g^{m+\theta}||_*||u_L^{m+\theta}||_a\right), \end{aligned} \tag{C.4}$$

Hence, with (C.3) and (C.4) and by the definition (C.1) of X^m,

$$||u_L^m||^2_{\mathcal{H}} - ||u_L^{m+1}||^2_{\mathcal{H}} \geq (2\theta - 1)||\vartheta||^2_{\mathcal{H}} - \quad 2k\left(||u_L^{m+\theta}||^2_a - ||g^{m+\theta}||_*||u_L^{m+\theta}||_a\right), \tag{C.5}$$

and adding $C_2 k||g^{m+\theta}||^2_* - C_1 k||u_L^{m+\theta}||^2_a$ on both sides yields

$$\begin{aligned} X^m \geq (2\theta - 1)||\vartheta||^2_{\mathcal{H}} - &\quad 2k\left(||u_L^{m+\theta}||^2_a - ||g^{m+\theta}||_*||u_L^{m+\theta}||_a\right) \\ + &\quad C_2 k||g^{m+\theta}||^2_* - C_1 k||u_L^{m+\theta}||^2_a. \end{aligned} \tag{C.6}$$

In case $\frac{1}{2} \leq \theta \leq 1$, the right hand side is positive whenever

$$0 < C_1 < 2, \qquad C_2 \geq \tfrac{1}{2 - C_1}. \tag{C.7}$$

In case $0 \leq \theta \leq \frac{1}{2}$ once again from the θ scheme (3.26)

$$(\vartheta, v_L)_{\mathcal{H}} = k(-Au_L^{m+\theta} + g^{m+\theta}, v_L)_{\mathcal{H}},$$

which now yields by the definition of the dual norm (3.24) and of λ_A the estimate

$$||\vartheta||_{\mathcal{H}} \leq \lambda_A^{1/2}||\vartheta||_* \leq \lambda_A^{1/2} k\left(||Au_L^{m+\theta}||_* + ||g^{m+\theta}||_*\right), \tag{C.8}$$

where the first term on the right hand side satisfies $||Au_L^{m+\theta}||_* = ||u_L^{m+\theta}||_a$. This equality holds by the estimates $(Au_L^{m+\theta}, v_L)_{\mathcal{H}} = a(u_L^{m+\theta}, v_L) \leq ||u_L^{m+\theta}||_a||v_L||_a$ and $||Au_L^{m+\theta}||_* \geq ||u_L^{m+\theta}||_a$, where the latter inequality is obtained by setting $v_L := u_L^{m+\theta}$, and the former inequality by polarisation[17]. Hence, from (C.8) we conclude

$$||\vartheta||^2_{\mathcal{H}} \leq \lambda_A k^2\left(||u_L^{m+\theta}||_a + ||g^{m+\theta}||_*\right)^2,$$

and therefore we proved $X^m \geq 0$, $m = 0, \ldots, M-1$, since the above together with (C.6) yields

$$X^m \geq k(2 - C1 - \sigma)||u_L^{m+\theta}||_a - k2(1+\sigma)||g^{m+\theta}||_*||u_L^{m+\theta}||_a + k(C_2 - \sigma)||g^{m+\theta}||_*.$$

□

Proof of Lemma 4.5. The statement of the Lemma is then is by the decomposition 4.17 essentially a consequence of the fact that the errors ξ satisfy the same PDE as the functions u: The variational formulation (2.11) implies

$$(\dot{u}^{m+\theta}, v)_{\mathcal{V}\times\mathcal{V}^*} + a(u^{m+\theta}, v) = (g^{m+\theta}, v)_{\mathcal{V}\times\mathcal{V}^*} \quad \forall v \in \mathcal{V}. \tag{C.9}$$

[17] For the square of the expression $a(u_L^{m+\theta}, v_L)$ there holds the inequality $a(u,v)^2 \leq \frac{1}{4}\left(a(u+v, u+v) + a(u-v, u-v)\right)^2 \leq \frac{1}{2}\left(a(u,u) + a(v,v)\right)^2 \leq a(u,u)a(v,v)$.

Using the definition (4.1) of ξ, we rewrite (4.15) in the form:

$$\begin{aligned}&\tfrac{1}{k}(\xi_L^{m+1}-\xi_L^m,v_L)_{\mathcal{V}\times\mathcal{V}^*}+a(\theta\xi_L^{m+1}+(1-\theta)\xi_L^m,v_L)=\\&\tfrac{1}{k}((P_Lu^{m+1}-u_L^{m+1})-(P_Lu^m-u_L^m),v_L)_{\mathcal{V}\times\mathcal{V}^*}+a(P_Lu^{m+\theta}+u_L^{m+\theta},v_L)=\\&\left(\tfrac{P_Lu^{m+1}-P_Lu^m}{k},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}+a(P_Lu^{m+\theta},v_L)-\left(\tfrac{1}{k}(u_L^{m+1}-u_L^m,v_L)_{\mathcal{V}\times\mathcal{V}^*}-a(u_L^{m+\theta},v_L)\right)\end{aligned}$$

where we used $u^{m+\theta}:=\theta u^{m+1}+(1-\theta)u^m$ and the linearity of the projector:

$$\theta P_Lu^{m+1}-(1-\theta)P_Lu_L^{m+1}=P_Lu^{m+\theta}.$$

Furthermore, by the θ-scheme (3.26) for u, and by (C.9)

$$\begin{aligned}&\left(\tfrac{P_Lu^{m+1}-P_Lu^m}{k},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}+a(P_Lu^{m+\theta},v_L)-\left(\left(\tfrac{u_L^{m+1}-u_L^m}{k},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}-a(u_L^{m+\theta},v_L)\right)\\&=\left(\tfrac{P_Lu^{m+1}-P_Lu^m}{k},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}+a(P_Lu^{m+\theta},v_L)-(g^{m+\theta},v_L)_{\mathcal{V}\times\mathcal{V}^*}\\&=\left(\tfrac{P_Lu^{m+1}-P_Lu^m}{k},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}+a(P_Lu^{m+\theta},v_L)-a(u^{m+\theta},v_L)-(\dot u^{m+\theta},v_L)_{\mathcal{V}\times\mathcal{V}^*}\\&=\left(\tfrac{P_Lu^{m+1}-P_Lu^m}{k}-\tfrac{u^{m+1}-u^m}{k},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}+a\left(P_Lu^{m+\theta}-u^{m+\theta},v_L\right)+\left(\tfrac{u^{m+1}-u^m}{k}-\dot u^{m+\theta},v_L\right)_{\mathcal{V}\times\mathcal{V}^*}\\&=(r_2,v_L)_{\mathcal{V}\times\mathcal{V}^*}+(r_3,v_L)_{\mathcal{V}\times\mathcal{V}^*}+(r_1,v_L)_{\mathcal{V}\times\mathcal{V}^*}.\end{aligned}$$

□

Proof of Lemma 4.7. We adapt [41, Section 3.6.2] and [60, Lemma 5.3] to the situation at hand and confirm that the estimates of [60, Lemma 5.3] carry over to the weighted case. Analogously to the classical (unweighted) case, the statement of the Lemma follows from $||r^m||_*^2\le||r_1^m||_*^2+||r_2^m||_*^2+||r_3^m||_*^2$ and the corresponding norm estimates for the decomposition (4.17). The estimate of the residual r_1 is

$$\begin{aligned}||r_1||_*&=||\tfrac{1}{k}(u^{m+1}-u^m)-\dot u^{m+\theta}||_*&&\le\tfrac{1}{k}\left(\int_{t_m}^{t_{m+1}}|s-(1-\theta)t_{m+1}-\theta t_m|\;||\ddot u||_*ds\right)\\&&&\le\tfrac{C_\theta}{k^{1/2}}\left(\int_{t_m}^{t_{m+1}}||\ddot u||_*^2ds\right)^{1/2}.\end{aligned}\tag{C.10}$$

In case $\theta=\frac{1}{2}$, partial integration yields the refined estimate

$$\begin{aligned}||r_1||_*&=||\tfrac{1}{k}(u^{m+1}-u^m)-\dot u^{m+\theta}||_*&&\le\tfrac{1}{2k}\left(\int_{t_m}^{t_{m+1}}|(t_{m+1}-s)(t_m-s)|\;||\dddot u||_*ds\right)\\&&&\le\tfrac{C}{k^{3/2}}\left(\int_{t_m}^{t_{m+1}}||\dddot u||_*^2ds\right)^{1/2}.\end{aligned}\tag{C.11}$$

The norm of the residual r_2 is bounded by

$$||r_2||_*\quad\le2^{-L}\tfrac{C}{k^{1/2}}\left(\int_{t_m}^{t_{m+1}}||\dot u||_{\mathcal{V}}^2ds\right)^{1/2}.\tag{C.12}$$

The bound (C.12) follows from (3.24) and from the estimate

$$\begin{aligned}|(r_2,v_L)_{\mathcal{V}\times\mathcal{V}^*}|&=|(\tfrac{1}{k}(P_Lu^{m+1}-P_Lu^m)+\tfrac{1}{k}(u^{m+1}-u^m),v_L)_{\mathcal{V}\times\mathcal{V}^*}|\\&\le C||\tfrac{1}{k}(P_Lu^{m+1}-P_Lu^m)+\tfrac{1}{k}(u^{m+1}-u^m)||_*||v_L||_a\\&\le\tfrac{C}{k}||(I-P_L)\int_{t_m}^{t_{m+1}}\dot u(s)\;ds||_*||v_L||_a\\&\le\tfrac{C}{k^{1/2}}\left(\int_{t_m}^{t_{m+1}}||\dot u-P_L\dot u||_{\mathcal{H}}^2ds\right)^{1/2}\;||v_L||_a\\&\overset{(4.4)}{\le}\tfrac{C}{k^{1/2}}(2^{-L})^{c_{\mathbf{a}}}\left(\int_{t_m}^{t_{m+1}}||\dot u||_{\mathcal{H}_j^1}^2ds\right)^{1/2}\;||v_L||_a,\end{aligned}\tag{C.13}$$

where the last step follows from the approximation property (4.8) resp. (4.9) and the estimate (A.16). The norm of the residual r_3 allows for the upper bound

$$||r_3||_*\quad\le C(2^{-L})^{c_{\mathbf{a}}}\;||u^{m+\theta}||_{\mathcal{H}_j^2}.\tag{C.14}$$

The estimate (C.14) follows from (3.24) and from

$$\begin{aligned}|(r_3,v_L)_{\mathcal{V}\times\mathcal{V}^*}|&=|a(P_Lu^{m+\theta}-u^{m+\theta},v_L)|\\&\le||P_Lu^{m+\theta}-u^{m+\theta}||_a||v_L||_a\overset{(4.4)}{\le}C(2^{-L})^{c_{\mathbf{a}}}\;||u^{m+\theta}||_{\mathcal{H}_j^2}||v_L||_a.\end{aligned}\tag{C.15}$$

The second step follows from a simple polarisation argument

$$|(r_3, v_L)_{\mathcal{V}\times\mathcal{V}^*}|^2 \quad \leq \left|\tfrac{1}{4}\left(a(u+v,u+v)+a(u-v,u-v)\right)\right|^2 \leq \left|\tfrac{1}{2}\left(a(u,u)+a(v,v)\right)\right|^2 \leq |(a(u,u)a(v,v))|$$

and in the last step we used $||\cdot||_a \approx ||\cdot||_{\mathcal{V}}$ (which holds by well-posedness, cf. Theorem 2.17 and equation (2.7)) and the approximation property (4.8) resp. (4.9).

□

C.1. **Approximation estimates for CEV: Alternative.** If we do not assume any additional integrability requirements on our solution up to first order derivatives, we stated in Remark 4.4 that we obtain such approximation estimates where the order of approximation depends on the parameter β. The proof of the estimates (4.9) in Remark 4.4 is similar to that of Proposition 4.3 and is included here:

Proof of Remark 4.4. For $j=1$ and $\mu \in [\max\{-1,-2\beta\}, 1-2\beta]$ the inequalities in (4.10) become

$$\begin{aligned}||(u-P_Lu)'||^2_{L^2(I,x^{\mu/2+\beta})} &= \tilde{C}\sum_{l=L+1}^{\infty} 2^{2l}\sum_{k=0}^{2^l}(2^{-l}k)^{\mu+2\beta}|u_k^l|^2\\ &\geq \tilde{C}2^{2L(1-\beta)}\sum_{l=0}^{\infty}\sum_{k=0}^{2^l}2^{2l(\beta-(\mu/2+\beta))}(k)^{\mu+2\beta}|u_k^l|^2 \qquad \text{(C.16)}\\ &\geq \tilde{C}2^{2L(1-\beta)}\sum_{l=0}^{\infty}\sum_{k=0}^{2^l}(2^{-l}k)^{\mu}|u_k^l|^2 = \tilde{C}2^{2L(1-\beta)}||u-P_Lu||^2_{L^2(I,x^{\mu/2})}.\end{aligned}$$

Similarly, (C.16) after replacing $(u-P_Lu)'$ by $(u-P_Lu)''$ and $(u-P_Lu)$ by $(u-P_Lu)'$ reads

$$\begin{aligned}||(u-P_Lu)''||^2_{L^2(I,x^{\mu/2+2\beta})} &= \tilde{C}\sum_{l=L+1}^{\infty} 2^{4l}\sum_{k=0}^{2^l}(2^{-l}k)^{\mu+4\beta}|u_k^l|^2\\ &\geq \tilde{C}2^{2L(1-\beta)}\sum_{l=0}^{\infty}2^{2l}\sum_{k=0}^{2^l}2^{2l(\beta-(\mu/2+2\beta))}(k)^{\mu+2\beta}|u_k^l|^2 \qquad \text{(C.17)}\\ &\geq \tilde{C}2^{2L(1-\beta)}\sum_{l=L+1}^{\infty}2^{2l}\sum_{k=0}^{2^l}(2^{-l}k)^{\mu+2\beta}|u_k^l|^2 = \tilde{C}2^{2L(1-\beta)}||(u-P_Lu)'||^2_{L^2(I,x^{\mu/2+\beta})}.\end{aligned}$$

Finally, combining (C.16) and (C.17) yields the last estimate in (4.9). □

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